II A famous example, the van der Pol oscillator

(1) The model

The Van-der-Pol oscillator is one of the most famous examples of a nonlinear oscillator to have a stable limit cycle.

It is used to model many systems in engineering (electric circuits, ...), biological systems (circadian clocks, neurons firing ..), chemistry, etc...

General form of the equation:

$$\frac{d^2 f}{dt^2} + f = \underbrace{\varepsilon(1 - f^2)}\frac{df}{dt}$$

note the nonlinear "damping" term.

- for small f, $\sim \varepsilon \frac{df}{dt}$
  → exponential amplification
- for large f $\sim -\varepsilon f^2 \frac{df}{dt}$
  → strong nonlinear damping.

→ We expect f to be amplified until its amplitude reaches about 1, then to stay at that amplitude because of competition between amplification & damping.

→ The oscillator is expected to have a limit cycle, the question is, can we determine what this cycle is?

From here on we assume as initial conditions $f(0) = h$

$$\frac{df}{dt}(0) = 0$$

(2) Multiscale analysis

let $T_0 = t$ and $T_1 = \epsilon t$

We saw last chapter that

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial T_0} + \epsilon \frac{\partial}{\partial T_1}$$

$$\frac{\partial^2}{\partial T^2} = \frac{\partial^2}{\partial T_0^2} + 2\epsilon \frac{\partial^2}{\partial T_0 \partial T_1} + \epsilon^2 \frac{\partial^2}{\partial T_1^2}$$

We let $f = f_0 + \epsilon f_1 + \cdots$

$\Rightarrow$

$$\left(\frac{\partial^2}{\partial T_0} + 2\epsilon \frac{\partial^2}{\partial T_1 \partial T_0} + \epsilon^2 \frac{\partial^2}{\partial T_1^2}\right)(f_0 + \epsilon f_1 + \cdots) + (f_0 + \epsilon f_1 + \cdots)$$

$$= \epsilon\left(1 - (f_0 + \epsilon f_1 + \cdots)^2\right)\left(\frac{\partial}{\partial T_0} + \epsilon \frac{\partial}{\partial T_1}\right)(f_0 + \epsilon f_1 + \cdots)$$

$$= \epsilon\left(1 - f_0^2 - 2\epsilon f_0 f_1 - \epsilon^2 (f_1^2 + 2 f_0 f_2 + \cdots)\right)\left(\frac{\partial f_0}{\partial T_0} + \epsilon\left(\frac{\partial f_1}{\partial T_0} + \frac{\partial f_0}{\partial T_1}\right) + \right.$$

So to 0th order we have:

$$\begin{cases} \dfrac{\partial^2 f_0}{\partial T_0^2} + f_0 = 0 \\ f_0(0,0) = h \\ \dfrac{\partial f_0}{\partial T_0}(0,0) = 0 \end{cases} \xrightarrow{\text{as before}} \quad f_0(T_0, T_1) = A_0(T_1)\cos T_0 + B_0(T_1)\sin T_0$$

with $\begin{cases} A_0(0) = h \\ B_0(0) = 0 \end{cases}$

to 1st order, we have

$$\begin{cases} \dfrac{\partial^2 f_1}{\partial T_0^2} + f_1 = -2\dfrac{\partial^2 f_0}{\partial T_0 \partial T_1} + (1 - f_0^2)\dfrac{\partial f_0}{\partial T_0} \\ f_1(0,0) = 0 \\ \dfrac{\partial f_1}{\partial T_0} + \dfrac{\partial f_0}{\partial T_1} = 0 \end{cases}$$

The equation itself has a RHS that reads

$$-2\left(-\frac{\partial A_0}{\partial T_1}\sin T_0 + \frac{\partial B_0}{\partial T_1}\cos T_0\right) + \left(1 - (A_0\cos T_0 + B_0 \sin T_0)^2\right)\left(-A_0 \sin T_0 + B_0 \cos T_0\right.$$

$$\Rightarrow \quad 2\left(\frac{\partial A_0}{\partial T_1}\sin T_0 - \frac{\partial B_0}{\partial T_1}\cos T_0\right)$$

$$+ \left(1 - A_0^2\cos^2 T_0 - 2A_0B_0\cos T_0 \sin T_0 - B_0^2 \sin^2 T_0\right)\left(B_0\cos T_0 - A_0 \sin T_0\right)$$

$$= 2\left(\frac{\partial A_0}{\partial T_1}\sin T_0 - \frac{\partial B_0}{\partial T_1}\cos T_0\right)$$

$$+ B_0\cos T_0 - A_0\sin T_0 - A_0^2 B_0 \cos^3 T_0 + A_0^3\cos^2 T_0 \sin T_0$$

$$- 2A_0B_0^2\cos^2 T_0 \sin T_0 + 2A_0^2 B_0 \cos T_0 \sin^2 T_0 + A_0 B_0^2 \sin^3 T_0 - B_0^3 \sin^2 T_0 \cos T_0$$

$$\cancel{= 2\left[\frac{\partial A_0}{\partial T_1}\sin T_0 - \frac{\partial B_0}{\partial T_1}\cos T_0\right] + B_0\cos T_0 - A_0\sin T_0}$$

$$\cancel{+ A_0^2 B_0 \cos^2 T_0 \sin T_0 - A_0 B_0^2 \cos T_0 \sin^2 T_0}$$

$$\cancel{+ A_0^3\cos^3 T_0 - B_0^3 \sin^3 T_0}$$

Now $\cos^2 T_0 \sin T_0 = \frac{1}{2}(1+\cos 2T_0)\sin T_0$

$$= \frac{1}{4}\sin T_0 + \frac{1}{4}\sin(3T_0)$$

similarly, $\cos T_0 \sin^2 T_0 = \frac{1}{4}(\cos T_0 - \cos 3T_0)$

$$\cos^3 T_0 = \frac{3}{4}\cos T_0 + \frac{1}{4}\cos 3T_0, \quad \sin^3 T_0 = \frac{3}{4}\sin T_0 - \frac{1}{4}\sin 3T_0$$

→ the terms that would lead to secular terms in f_0 are:

$$2\left[\frac{\partial A_0}{\partial T_1}\sin T_0 - \frac{\partial B_0}{\partial T_1}\cos T_0\right] + B_0\cos T_0 - A_0\sin T_0 - \frac{3}{4}A_0^2 B_0\cos T_0$$

$$+ \frac{A_0^3}{4}\sin T_0 - \frac{A_0B_0^2}{2}\sin T_0 + \frac{A_0^2 B_0}{2}\cos T_0 + \frac{3}{4}A_0B_0^2\sin T_0 - \frac{B_0^3}{4}\cos T_0$$

To eliminate them, let

$$\left\{\begin{array}{ll} 2\dfrac{\partial A_0}{\partial T_1} - A_0 + \dfrac{A_0^3}{4} + \dfrac{A_0B_0^2}{4} = 0 & (1) \\ -2\dfrac{\partial B_0}{\partial T_1} + B_0 - \dfrac{A_0^2B_0}{4} - \dfrac{B_0^3}{4} = 0 & (2) \end{array}\right\} \text{The compatibility condition for } A_0(T_1) \text{ and } B_0(T_1)$$

→ 2 coupled nonlinear equations ... this seems untractable !?

In fact, there is a much better way of dealing with the problem that will lead to an analytical solution without having to use all these nasty trig identities.

Idea: use the exponential notation for f_0.

we had $\quad f_0(T_0, T_1) = A_0(T_1)\cos T_0 + B_0(T_1)\sin T_0$

$\rightarrow$ rewrite as $\quad f_0(T_0, T_1) = A_0(T_1)e^{iT_0} + A_0^*(T_1)e^{-iT_0}$

$$= \mathrm{Re}\left(A_0(T_1)e^{iT_0}\right)$$

The RHS of the equation for f_1 then becomes:

$$-2\frac{\partial^2 f_0}{\partial T_0 \partial T_1} + (1-f_0^2)\frac{\partial f_0}{\partial T_0}$$

$$= -2\left[i\frac{\partial A_0}{\partial T_1}e^{iT_0} - i\frac{\partial A_0^*}{\partial T_1}e^{-iT_0}\right]$$

$$+\left[1 - A_0^2 e^{2iT_0} - 2A_0A_0^* - A_0^{*2}e^{-2iT_0}\right]\left(iA_0e^{iT_0} - iA_0^*e^{-iT_0}\right)$$

$$= -2i\left[\frac{\partial A_0}{\partial T_1}e^{iT_0} - \frac{\partial A_0^*}{\partial T_1}e^{-iT_0}\right] + i\left(1-2A_0A_0^*\right)\left(A_0e^{iT_0} - A_0^*e^{-iT_0}\right)$$

$$-iA_0^3e^{3iT_0} + iA_0^2A_0^*e^{iT_0} - iA_0^{*2}A_0e^{-iT_0} + A_0^{*3}e^{-3iT_0}$$

To eliminate all source of secular term, we simply need to set:

$$\begin{cases} -2i\dfrac{\partial A_0}{\partial T_1} + iA_0(1-2A_0A_0^*) + iA_0^2A_0^* = 0 \\[2ex] 2i\dfrac{\partial A_0^*}{\partial T_1} - iA_0^*(1-2A_0A_0^*) - iA_0^{*2}A_0 = 0 \end{cases}$$

Since $A_0A_0^* = |A_0|^2$, we recognize the second equation as the c.c. of the first. The latter simplifies as:

$$-2\frac{\partial A_0}{\partial T_1} + A_0 - A_0^2A_0^* = 0$$

$$\Rightarrow \quad \frac{\partial A_0}{\partial T_1} = \frac{1}{2}A_0\left(1-|A_0|^2\right)$$

This, incidentally, is also a set of coupled nonlinear ODEs (say, one for $Re(A_0)$ and one for $Im(A_0)$), but if we write $A_0 = |A_0|e^{i\theta}$ (norm-argument form) then

$$\frac{\partial}{\partial T_1}\left(|A_0|e^{i\theta}\right) = \frac{1}{2}|A_0|e^{i\theta}\left(1-|A_0|^2\right)$$

$$\rightarrow \quad \frac{\partial |A_0|}{\partial T_1}e^{i\theta} + i\frac{\partial \theta}{\partial T_1}|A_0| = \frac{1}{2}|A_0|e^{i\theta}\left(1-|A_0|^2\right)$$

we see that a solution can be obtained if

$$\begin{cases} \dfrac{\partial |A_0|}{\partial T_1} = \dfrac{1}{2}|A_0|\left(1-|A_0|^2\right) & \leftarrow \text{this is now a } \underline{\text{single}} \text{ nonlinear ODE} \\ \dfrac{\partial \theta}{\partial T_1} = 0 & \leftarrow \text{this has the simple solution } \theta = \text{constant} \end{cases}$$

To solve the equation for the norm, note that it is separable: $\Rightarrow$

$$\frac{d|A_0|}{|A_0|\left(1-|A_0|^2\right)} = \frac{dT_1}{2}$$

Using integral tables (or otherwise) we get

$$\ln\left|\sqrt{\frac{|A_0|^2}{1-|A_0|^2}}\right| = \frac{T_1}{2} + C \quad \leftarrow C \text{ obtained from initial conditions}$$

- Since, at $T_1 = 0$, we need $f_0(0,0) = h$, that is

  $A_0(0)e^0 + A_0^*(0)e^0 = h \Rightarrow 2Re\,A_0 = h$ so

  $|A_0| = \frac{h}{2}$ at time $T_1 = 0$

- with $\frac{\partial f_0}{\partial T_0} = 0 \rightarrow iA_0(0)e^0 - iA_0^*(0)e^0 = 0$

  $\Rightarrow A_0(0) = A_0^*(0) \Rightarrow A_0$ is real.

This implies: $\begin{cases} \theta(0) = 0 \text{ and} \\ |A_0|(0) = \frac{h}{2} \end{cases}$

Plugging this in, we get:

$$\frac{1}{2}\ln\left|\frac{\frac{h^2}{4}}{1-\frac{h^2}{4}}\right| = 0 + C \quad \Rightarrow C = \ln\sqrt{\frac{h^2}{|4-h^2|}}$$

So
$$\ln\sqrt{\frac{|A_0|^2}{|1-|A_0|^2|}} = \frac{T_1}{2} + \ln\sqrt{\frac{h^2}{|4-h^2|}}$$

$$\Rightarrow \quad \frac{|A_0|^2}{|1-|A_0|^2|} = e^{T_1}\cdot\frac{h^2}{|4-h^2|}$$

This leaves 2 possibilities:

- if $|A_0|<1$ then:

$$\frac{|A_0|^2}{1-|A_0|^2} = e^{T_1}\frac{h^2}{|4-h^2|}$$

$$\Rightarrow \quad |A_0|^2 = \frac{e^{T_1}\frac{h^2}{|4-h^2|}}{1+e^{T_1}\frac{h^2}{|4-h^2|}} = \frac{1}{1+\frac{|4-h^2|}{h^2}e^{-T_1}}$$ which is always <1

- if $|A_0|>1$ then:

$$\frac{|A_0|^2}{|A_0|^2-1} = e^{T_1}\frac{h^2}{|4-h^2|}$$

$$\Rightarrow \quad |A_0|^2 = \frac{1}{1-\frac{|4-h^2|}{h^2}e^{-T_1}}$$ which is always >1.

So finally, we get that: A_0 is real, and $|A_0|$ is given above

$$\Rightarrow \quad f_0(T_0,T_1) = A_0(T_1)e^{iT_0} + A_0^*(T_1)e^{-iT_0}$$
$$= A_0(T_1)\left(e^{iT_0}+e^{-iT_0}\right)$$
$$= 2A_0(T_1)\cos T_0 = \frac{2\cos T_0}{\sqrt{1\pm\frac{|4-h^2|}{h^2}e^{-T_1}}} \Rightarrow$$

The correct multiscale expansion for $f(t)$ is (to lowest order in ε),

$$f(t) = \frac{2\cos t}{\sqrt{1 \pm \frac{|4-h^2|}{h^2} e^{-\varepsilon t}}}$$

where $\pm$ depends on the size of A_0.

Note that, as $t \to \infty$, $f(t) \to 2\cos t$.

This is not entirely intuitive: based on the simple argument given in the introduction, we might have expected to have $f(t) \to \cos t$ instead...

However, this is indeed the correct solution. Its physical interpretation is that the oscillator spends some time of the period being amplified (when $f(t) < 1$) and some time being damped (when $f(t) > 1$), resulting in an overall stable oscillation.

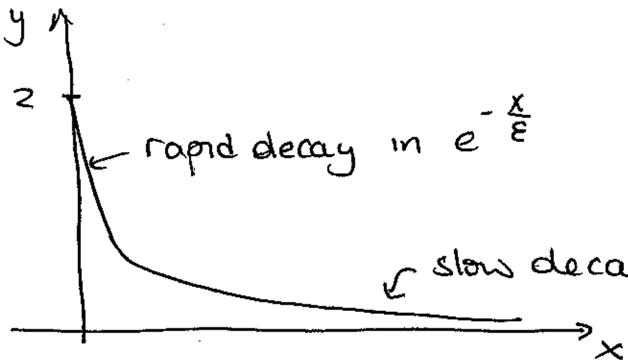
The multi-scale expansion doesn't just apply to oscillators. We now study 2 more examples that also exhibit multi-scale properties without being oscillators.

III Examples of multi-scale analysis for singular problems

① Recall the problem from Chapter I:

$$\begin{cases} \varepsilon \dfrac{dy}{dx} + y = e^{-x} \\ y(0) = 2 \end{cases}$$

We saw that this problem had a bounday layer;



In this case, the multiscale method also works. This time our "fast variable" is $X_0 = \frac{x}{\varepsilon}$ while our "slow" variable is $X_1 = x$

In that case $\dfrac{d}{dx} = \dfrac{dX_0}{dx}\dfrac{\partial}{\partial X_0} + \dfrac{dX_1}{dx}\dfrac{\partial}{\partial X_1}$

$$= \frac{1}{\varepsilon}\frac{\partial}{\partial X_0} + \frac{\partial}{\partial X_1}$$

$\Rightarrow$ Let $y = y_0 + \varepsilon y_1 + \cdots$

Then we have

$$\begin{cases} \varepsilon\left(\frac{1}{\varepsilon}\frac{\partial}{\partial X_0} + \frac{\partial}{\partial X_1}\right)(y_0 + \varepsilon y_1 + \cdots) + (y_0 + \varepsilon y_1 + \cdots) = e^{-X_1} \\ y(0,0) = 2 \end{cases}$$

$\rightarrow$ To zeroth order in ε we get

$$\begin{cases} \dfrac{\partial y_0}{\partial X_0} + y_0 = e^{-X_1} \\ y_0(0,0) = 2 \end{cases}$$

Since this is a partial differential equation, X_1 is "fixed" when we vary X_0. The solution is therefore

$$y_0(X_0, X_1) = e^{-X_1} + A_0(X_1)e^{-X_0}$$

where the amplitude $A(X_1)$ remains TBD, but satisfies $y_0(0,0) = 2 = 1 + A_0(0) \Rightarrow A_0(0) = 1$

To the next order we have

$$\begin{cases} \dfrac{\partial y_0}{\partial X_1} + \dfrac{\partial y_1}{\partial X_0} + y_1 = 0 \\ y_1(0,0) = 0 \end{cases}$$

$$\Rightarrow \frac{\partial y_1}{\partial X_0} + y_1 = -\left(-e^{-X_1} + \frac{\partial A_0}{\partial X_1} e^{-X_0}\right)$$

$$= e^{-X_1} + \frac{\partial A_0}{\partial X_1} e^{-X_0}$$

To satisfy the compatibility condition, we must not have any term on the RHS that could lead to secular terms in $X_0 \Rightarrow$ we need $\dfrac{\partial A_0}{\partial X_1} = 0$

This then implies $A_0(X_1) = \text{const} = A_0(0) = 1$

So finally, to the lowest order,

$$y(x) \cong y_0(X_0, X_1) = e^{-X_0} + e^{-X_1}$$

$$\cong e^{-\frac{x}{\varepsilon}} + e^{-x} + \text{h.o.t}$$

Comparing this with the exact solution we had

$$y(x) = \frac{1-2\varepsilon}{1-\varepsilon} e^{-\frac{x}{\varepsilon}} + \frac{e^{-x}}{1-\varepsilon}$$

we see that $y(x) = e^{-\frac{x}{\varepsilon}} + e^{-x}$ is indeed the correct uniform expansion at this order!

② Another example, this time a 2pt BVP

This method also works for some boundary-value problems, but this time with a lot of care

Consider
$$\begin{cases} \varepsilon \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = 0 \\ y(0) = 0 \\ y(1) = 1 \end{cases}$$

Suppose that (because of the ε on the highest-derivative) we have as before $X_0 = \frac{x}{\varepsilon}$, $X_1 = x$

Then
$$\frac{d^2}{dx^2} = \frac{1}{\varepsilon^2}\frac{\partial^2}{\partial X_0^2} + \frac{2}{\varepsilon}\frac{\partial^2}{\partial X_0 \partial X_1} + \frac{\partial^2}{\partial X_1^2}$$

$\Rightarrow$ let $y = y_0 + \varepsilon y_1 + \cdots$ as usual, then

$$\begin{cases} \varepsilon\left(\frac{1}{\varepsilon^2}\frac{\partial^2}{\partial X_0^2} + \frac{2}{\varepsilon}\frac{\partial^2}{\partial X_0\partial X_1} + \frac{\partial^2}{\partial X_1^2}\right)(y_0 + \varepsilon y_1 + \cdots) \\ \quad + \left(\frac{1}{\varepsilon}\frac{\partial}{\partial X_0} + \frac{\partial}{\partial X_1}\right)(y_0 + \varepsilon y_1 + \cdots) + (y_0 + \varepsilon y_1 + \cdots) = 0 \\ y_0(0,0) = 0 \qquad y_1(0,0) = 0 \quad \cdots \\ y_0(\frac{1}{\varepsilon}, 1) = 1 \qquad y_1(\frac{1}{\varepsilon}, 1) = 0 \quad \cdots \end{cases}$$

since $X_0 = \frac{1}{\varepsilon}$ when $x = 1$
and $X_1 = 1$ " " "

$\Rightarrow$ To order ε^{-1}:

$$\begin{cases} \dfrac{\partial^2 y_0}{\partial X_0} + \dfrac{\partial y_0}{\partial X_0} = 0 & \to \text{has solutions } y_0(X_0, X_1) = A(X_1)e^{-X_0} + B(X_1) \\ y_0(0,0) = 0 & \longrightarrow \text{implies: } A(0) + B(0) = 0 \\ y_0(\frac{1}{\varepsilon}, 1) = 1 & \longrightarrow \text{implies: } A(1)e^{-\frac{1}{\varepsilon}} + B(1) = 1 \end{cases}$$

To zeroth order

$$\begin{cases} \dfrac{\partial^2 y_1}{\partial X_0^2} + 2\dfrac{\partial^2 y_0}{\partial X_0 \partial X_1} + \dfrac{\partial y_1}{\partial X_0} + \dfrac{\partial y_0}{\partial X_1} + y_0 = 0 \\ y_1(0,0) = 0 \\ y_1(\frac{1}{\varepsilon}, 1) = 0 \end{cases}$$

$$\Rightarrow \frac{\partial^2 y_1}{\partial X_0^2} + \frac{\partial y_1}{\partial X_0} = -\left[2\frac{\partial^2 y_0}{\partial X_0 \partial X_1} + \frac{\partial y_0}{\partial X_1} + y_0\right]$$

$$= -\left[-2\frac{\partial A}{\partial X_1} e^{-X_0} + \frac{\partial A}{\partial X_1} e^{-X_0} + \frac{\partial B}{\partial X_1} + A e^{-X_0} + B\right]$$

To elimenate all tems that could lead to secular growth in X_0, we need to eliminate the tems in e^{-X_0} and the terms independent of X_0:

$$\begin{cases} -2\frac{\partial A}{\partial X_1} + \frac{\partial A}{\partial X_1} + A = 0 & \Rightarrow \frac{\partial A}{\partial X_1} = A \\ \frac{\partial B}{\partial X_1} + B = 0 & \Rightarrow \frac{\partial B}{\partial X_1} = -B \end{cases}$$

so $A(X_1) = A(0)e^{X_1}$ $\quad B(X_1) = B(0)e^{-X_1}$

Applying the BCs we then have:

$$A(0) + B(0) = 0$$
$$A(0)e^{1-\frac{1}{\varepsilon}} + B(0)e^{-1} = 1$$

$$\Rightarrow A(0) = -B(0)$$

$$\Rightarrow B(0)\left[e^{-1} - e^{1-\frac{1}{\varepsilon}}\right] = 1$$

$$\rightarrow B(0) = \frac{1}{e^{-1} - e^{1-\frac{1}{\varepsilon}}} = \frac{e}{1 - e^{2-1/\varepsilon}}$$

and finally, $y_0(X_0, X_1) = \frac{e}{e^{2-1/\varepsilon} - 1}\left[e^{X_1 - X_0} - e^{-X_1}\right]$

$$= \frac{e}{e^{2-1/\varepsilon} - 1}\left[e^{X - \frac{X}{\varepsilon}} - e^{-X}\right]$$

For small ε, $e^{2-\frac{1}{\varepsilon}} \simeq e^{-\frac{1}{\varepsilon}} \to 0$ so

$$\boxed{y(x) \to -e^{1+x-\frac{x}{\varepsilon}} + e^{1-x}}$$

This is indeed a uniform expansion of the true solution for small ε.