(4) Application to non-periodic systems

As discussed in the introduction, the power of the Lighthill / renormalization methods is that they can also be used for non-periodic systems (by contrast with the Lindsted – Poincaré technique which cannot).

We now see an example of how that may work.

Consider the equation

$$\begin{cases} (x+\varepsilon f)\dfrac{df}{dx} + f = 0 & 0 < \varepsilon \ll 1 \\ f(1) = 1 . \end{cases}$$

Before we proceed with the perturbation expansion, note that an exact solution to this equation actually exists. Let's see what it is, to understand what is about to happen.

$(x+\varepsilon f)\dfrac{df}{dx} + f = 0$ can be rewritten as

$\left(1+\varepsilon\dfrac{f}{x}\right)\dfrac{df}{dx} + \dfrac{f}{x} = 0$ → this is now a homogeneous form

To find a solution, let $g = \dfrac{f}{x}$ then $f = xg$ so

$$\frac{df}{dx} = x\frac{dg}{dx} + g$$

$$\Rightarrow \left(1+\varepsilon g\right)\left(g + x\frac{dg}{dx}\right) + g = 0 \Rightarrow x\frac{dg}{dx} = -\frac{g}{1+\varepsilon g} - g$$

$$= -\frac{g + (1+\varepsilon g)g}{1+\varepsilon g}$$

so $$dg \cdot \frac{(1+\varepsilon g)}{2g+\varepsilon g^2} = -\frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\ln\left(g + \frac{\varepsilon g^2}{2}\right) = -\ln|x| + \text{const}$$

$$\Rightarrow \quad g + \varepsilon \frac{g^2}{2} = \frac{K}{x^2}$$

$$\Rightarrow \quad \varepsilon \frac{g^2}{2} + g - \frac{K}{x^2} = 0$$

$$\Rightarrow \quad g = \frac{-1 \pm \sqrt{1 + 2K\varepsilon/x^2}}{\varepsilon}$$

$$\Rightarrow \quad f(x) = \frac{-x \pm \sqrt{x^2 + 2K\varepsilon}}{\varepsilon}$$

To have $f(1) = 1$, $\quad 1 = \dfrac{-1 \pm \sqrt{1 + 2K\varepsilon}}{\varepsilon}$

$\rightarrow$ this cannot happen for the $-$ solution (assuming $\varepsilon > 0$), so we must keep only the $+$ solution. Then, we need

$$\varepsilon = -1 + \sqrt{1 + 2K\varepsilon} \Rightarrow (\varepsilon + 1)^2 = 1 + 2K\varepsilon$$

$$\Rightarrow \quad K = \frac{(\varepsilon+1)^2 - 1}{2\varepsilon} = \frac{\varepsilon^2 + 2\varepsilon}{2\varepsilon} = \frac{\varepsilon + 2}{2}$$

And finally, $\quad f(x) = \dfrac{-x + \sqrt{x^2 + \varepsilon(\varepsilon+2)}}{\varepsilon}$

Note how:
- for $x = 0$, $f(0)$ is well-defined and equal to $f(0) = \sqrt{\dfrac{\varepsilon+2}{\varepsilon}}$
- for $x \to \pm\infty$, $f(x) \simeq \dfrac{-x + |x|\sqrt{1 + \varepsilon(\varepsilon+2)/x^2}}{\varepsilon}$

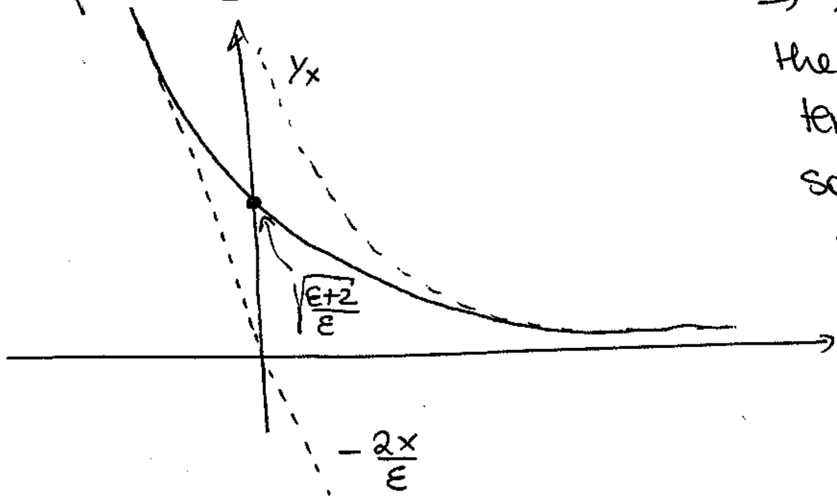
$$\simeq \left\{ -x + |x| \left(1 + \frac{\varepsilon(\varepsilon+2)}{2x^2} \right) \right\} \frac{1}{\varepsilon}$$

$$= -\frac{x}{\varepsilon} + \frac{|x|}{\varepsilon} + \frac{|x|(\varepsilon+2)}{2x^2} + \cdots$$

So if $x > 0$ then $\quad f(x) \simeq \dfrac{\varepsilon+2}{2x} + \cdots$

if $x < 0$ then $\quad f(x) \simeq -\dfrac{2x}{\varepsilon} - \dfrac{\varepsilon+2}{2x}$

Graphically:



→ As $\varepsilon \to 0$, $f(0) \to +\infty$, the asymptote as $x \to -\infty$ tends to the vertical axis, so the function $f(x)$ for $\varepsilon = 0$ is singular (and in fact, has the solution $f_s(x) = \frac{1}{x}$

Check: for $\varepsilon = 0$, $x\frac{df_s}{dx} + f_s = 0 \Rightarrow \frac{df_s}{dx} = -\frac{f_s}{x}$

$\rightarrow f_s(x) = \frac{K}{x}$; $f_s(1) = 1 \Rightarrow K = 1 \rightarrow f_s(x) = \frac{1}{x}$ ✓

⇒ In this particular problem, the case with $\varepsilon \neq 0$ ($\varepsilon > 0$) is a well-behaved non-singular equation, while $\varepsilon = 0$ is singular.

Let's now see what a standard perturbation technique yields. We begin by naively assuming an expansion of the form

$$f(x;\varepsilon) = f_0(x) + \varepsilon f_1(x) + \cdots$$

Plugging this in:

$$\left[x + \varepsilon(f_0 + \varepsilon f_1 + \cdots)\right]\left[f_0' + \varepsilon f_1' + \cdots\right] + (f_0 + \varepsilon f_1 + \cdots) = 0$$

⇒ to 0th order:

$$x f_0' + f_0 = 0 \qquad f_0(1) = 1$$

$$\Rightarrow f_0(x) = \frac{1}{x} \quad \text{as described earlier.}$$

⇒ To 1st order:

$$x f_1' + f_0 f_0' + f_1 = 0 \qquad f_1(1) = 0$$

$$\Rightarrow \quad x f_1' + f_1 = - f_0 f_0' = \frac{1}{x^3}$$

$$\Rightarrow \quad (x f_1)' = \frac{1}{x^3} \quad \Rightarrow \quad x f_1 = -\frac{1}{2x^2} + K_1$$

$$\text{for } f_1(1) = 0 \quad \Rightarrow \quad 0 = -\frac{1}{2} + K_1 \Rightarrow K_1 = \frac{1}{2}$$

$$\text{So} \quad f_1(x) = \frac{1}{2x}\left(1 - \frac{1}{x^2}\right)$$

$\Rightarrow$ To 2nd order,

$$x f_2' + f_2 + f_1 f_0' + f_0 f_1' \qquad \text{with } f_2(1) = 0$$

$$\Rightarrow \quad (f_2 x)' = -(f_1 f_0)'$$

$$\Rightarrow \quad f_2 x = - f_1 f_0 + K_2 = -\frac{1}{2x^2}\left(1 - \frac{1}{x^2}\right) + K_2$$

$$f_2(1) = 0 \Rightarrow \qquad \text{so } f_2(x) = -\frac{1}{2x^3}\left(1 - \frac{1}{x^2}\right)$$

So finally,
$$f(x) \simeq \frac{1}{x} + \frac{\varepsilon}{2x}\left(1 - \frac{1}{x^2}\right) - \frac{\varepsilon^2}{2x^3}\left(1 - \frac{1}{x^2}\right) + \dots$$
$$\simeq \frac{1}{x}\left[1 + \frac{\varepsilon}{2}\left(1 - \frac{1}{x^2}\right) - \frac{\varepsilon^2}{2x^2}\left(1 - \frac{1}{x^2}\right) + \dots\right]$$

It's quite clear that this is a non-uniform expansion as $x \to 0$. Indeed, the remainders cannot be bounded in an x-independent way, and the singularity increases with the order of expansion: $f_0 \sim O\left(\frac{1}{x}\right)$ $f_1 \sim O\left(\frac{1}{x^3}\right)$ $f_2 \sim O\left(\frac{1}{x^5}\right) \dots$

Let's now try to apply renormalization on this non-uniform expansion to try to see if we can remove the problem.

$$\text{Let} \quad x = s + \varepsilon a_1(s) + \varepsilon^2 a_2(s) + \dots$$

Then

$$f(s) \simeq \frac{1}{s+\varepsilon a_1(s)+\dots}\Big[1+\frac{\varepsilon}{2}\Big(1-\frac{1}{(s+\varepsilon a_1(s)+\dots)^2}\Big) - \frac{\varepsilon^2}{2(s+a_1(s)+\dots)^2}\Big(1-\frac{1}{(s+\varepsilon a_1(s)+\dots)^2}\Big)\Big]$$

$$\simeq \frac{1}{s\left[1+\frac{\varepsilon}{s}a_1(s)+\dots\right]}\left[\dots\right]$$

$$\simeq \frac{1}{s}\left(1-\frac{\varepsilon}{s}a_1(s)+\frac{\varepsilon^2}{s^2}a_1^2(s)-\frac{\varepsilon^2}{s}a_2(s)\right)\left[\dots\right]$$

to lowest orders in ε:

$$f(s) \simeq \frac{1}{s}+\varepsilon\left[-\frac{a_1(s)}{s^2}+\frac{1}{s}\left(\frac{1}{2}\left(1-\frac{1}{s^2}\right)\right)\right]$$
$$+\varepsilon^2\left[\frac{a_1^2}{s^3}-\frac{a_2}{s^2}-\frac{a_1}{s^2}\left(\frac{1}{2}\left(1-\frac{1}{s^2}\right)\right) + \frac{1}{s}\left(+\frac{1}{2}\frac{2a_1}{s^3}-\frac{1}{2s^2}\left(1-\frac{1}{s^2}\right)\right)\right]$$

$$\simeq \frac{1}{s}+\varepsilon\left[-\frac{a_1(s)}{s^2}+\frac{1}{2s}\left(1-\frac{1}{s^2}\right)\right]$$
$$+\varepsilon^2\left[-\frac{a_2}{s^2}+\frac{a_1^2}{s^3}-\frac{a_1}{2s^2}+\frac{3a_1}{2s^4}-\frac{1}{2s^3}\left(1-\frac{1}{s^2}\right)\right].$$

The fundamental singularity is in $\frac{1}{s}$ $\rightarrow$ we cannot hope to remove it, it's part of the solution. However, we can then choose all of the $a_1, a_2 \dots$ functions so that the <u>next</u> orders are no more singular than this first one $\rightarrow$

- ~~in~~ the $O(\varepsilon)$ equation we want to eliminate the terms in $\frac{1}{s^2}$ and $\frac{1}{s^3}$ $\rightarrow$ let's choose

$$\frac{a_1}{s^2} = -\frac{1}{2s^3} \Rightarrow a_1 = -\frac{1}{2s}$$

- in the $O(\varepsilon^2)$ equation, there are only terms in $\frac{1}{s^2}, \frac{1}{s^3}$, $\frac{1}{s^4}$ and $\frac{1}{s^5}$ → we need to eliminate all of them!

→ let
$$\frac{a_2}{s^2} = \frac{a_1^2}{s^3} - \frac{a_1}{2s^2} + \frac{3}{2}\frac{a_1}{s^4} - \frac{1}{2s^3}\left(1-\frac{1}{s^2}\right)$$
$$= \frac{1}{4s^3} - \frac{1}{2s^3} = -\frac{1}{4s^3}$$

$$\rightarrow \quad a_2 = -\frac{1}{4s}$$

So finally,
$$\begin{cases} x = s - \dfrac{\varepsilon}{2s} - \dfrac{\varepsilon^2}{4s} + o(\varepsilon^3) \\ f(s) = \dfrac{1}{s} + \varepsilon\left(\dfrac{1}{2s}\right) + o(\varepsilon^3) \end{cases}$$

To find out s in terms of x, let's solve for it:
$$s^2 - xs - \frac{\varepsilon}{2}\left(1+\frac{\varepsilon}{2}\right) = 0$$
$$s = \frac{x \pm \sqrt{x^2 + 2\varepsilon\left(1+\frac{\varepsilon}{2}\right)}}{2} + h.o.t$$
(select + root to have $x>0$ when $s>0$)

$$\Rightarrow \quad f(x) \cong \left(1+\frac{\varepsilon}{2}\right) \cdot \frac{2}{x + \sqrt{x^2 + 2\varepsilon\left(1+\frac{\varepsilon}{2}\right)}}$$

<u>Check</u> we now have

- $f(0) = \sqrt{\frac{2}{\varepsilon}} \cdot \sqrt{1+\frac{\varepsilon}{2}}$ to lowest order in ε,
  which is no longer singular but instead fairly close to the right solution

- When $x \to -\infty$, $|x| = -x$
$$f(x) \simeq \left(1+\frac{\varepsilon}{2}\right)\frac{2}{x} \cdot \frac{1}{1 - \sqrt{1 + \frac{2\varepsilon(1+\varepsilon/2)}{x^2}}}$$
$$\simeq \left(1+\frac{\varepsilon}{2}\right)\frac{2}{x} \cdot \frac{1}{1 - 1 - \frac{\varepsilon\left(1+\frac{\varepsilon}{2}\right)}{x^2}} \simeq -\frac{2x}{\varepsilon} \quad \text{as required}$$

We therefore see that $f(x) = \dfrac{2(1+\frac{\varepsilon}{2})}{x+\sqrt{x^2+2\varepsilon(1+\frac{\varepsilon}{2})}}$

is indeed a uniform expansion for $f(x)$, valid in all limits of interest.

Note: The choices of $a_1(s)$, $a_2(s)$, etc are not unique: you can always add a term that is of the same order of singularity as the one you need to keep without changing the uniformity of the expansion. The trick, therefore, is to choose these functions to make the solution as simple as possible.

5 The failure of the method of strained coordinates

It is important to note that the various method learned in this Chapter do not necessarily always work. One should therefore try to apply them, and if they fail, then move to a different set of tools.

A common example for the failure of the method of strained coordinates arises in periodic systems where the nonlinearity / perturbation induces a modulation in the amplitude of oscillation that proceeds on a different timescale from the oscillation period. This occurs, for instance, in the van der Pol oscillator:

$$\frac{d^2f}{dt^2} + f = \varepsilon(1-f^2)\frac{df}{dt}.$$

let's try to apply the techiques learned here, with $f(t)$ satisfying $f(0)=1$, $\frac{df}{dt}(0)=0$.

First assume $f(t) = f_0(t) + \varepsilon f_1(t) + \varepsilon^2 f_2(t)$

$\Rightarrow$ to 0th order

$$f_0'' + f_0 = 0 \qquad f_0(0)=1 \qquad \frac{df}{dt}(0)=0$$

$$\Rightarrow f_0(t) = \cos t$$

$\Rightarrow$ to 1st order

$$f_1'' + f_1 = (1-f_0^2)\frac{df_0}{dt} = (1-\cos^2 t)(-\sin t)$$

$$= -\sin^3 t = \frac{1}{4}\sin 3t - \frac{3}{4}\sin t$$

$$\rightarrow f_1(t) = A\cos t + B\sin t + k_1 t\cos t + k_2 \sin 3t$$

K_1 & K_2 satisfy

$$-9K_2 + K_2 = \frac{1}{4} \Rightarrow K_2 = -\frac{1}{32}$$

$$-2K_1 = -\frac{3}{4} \Rightarrow K_1 = \frac{3}{8}$$

To satisfy the IC $f_1(0)=0$ $\frac{df_1}{dt}(0)=0$, we then have to have

$$\begin{cases} A = 0 \\ B + \frac{3}{8} - \frac{3}{32} = 0 \end{cases} \rightarrow B = -\frac{9}{32}$$

so $f_1(t) = -\frac{9}{32}\sin t + \frac{3}{8}t\cos t - \frac{1}{32}\sin 3t$

and so $f(t) = \cos t + \varepsilon\left(\frac{3}{8}t\cos t - \frac{9}{32}\sin t - \frac{1}{32}\sin 3t\right)$

So far, everything seems to be as usual. Let's try to get a uniform 1-term expansion for $f(t)$, using renormalization let $\quad t = \tau + \varepsilon\omega_1(\tau) + \varepsilon^2\omega_2(\tau) + \cdots$

$$f(\tau) = \cos(\tau + \varepsilon\omega_1(\tau) + \cdots) + \varepsilon\underbrace{\left(\frac{3}{8}\tau\cos\tau - \frac{9}{32}\sin\tau - \frac{1}{32}\sin 3\tau\right)}$$

ignore the ε part in t
since we will only need
an expansion up to $O(\varepsilon)$
to get ω_1

$$= \cos\tau - \varepsilon\omega_1(\tau)\sin\tau + \varepsilon\left(\frac{3}{8}\tau\cos\tau - \frac{9}{32}\sin\tau - \frac{1}{32}\sin 3\tau\right)$$

To remove the secular term, we would need

$$\omega_1(\tau) = \frac{3}{8}\tau\frac{\cos\tau}{\sin\tau}$$

The problem is that this function is not defined for all values of τ → it is singular whenever $\tau = n\pi$. Also, it is multiply defined (not "one-to-one") so when writing $t = \tau + \varepsilon\omega_1(\tau)$, τ is not invertible for all t.

This example is fairly typical of the failure of the strained coordinates technique. In the next chapter, we will now study a very different approach which can be used to understand the dynamics of the vander Pol oscillator.