So the series is uniformly convergent, and is a uniform asymptotic expansion.

(4) Typical sources of non-uniformity in asymptotic expansions

There are two typical sources of non-uniformity in an expansion

- infinite domains, which allow "large values" of the independent variable to affect the convergence of the series (as in Chapter 1)
- Singularities in the governing equations as $\varepsilon \to 0$ (eg. existence of singular expansions)

Example 1: Infinite domain

Let's consider a nonlinear oscillator of the form

$$\frac{d^2 f}{dt^2} = -f(1+\varepsilon f^2) \quad \leftarrow \text{the Duffing equation.}$$

This equation commonly arises in mechanical & electrical systems (See later for more detail).

Let's assume for the moment that $f(0) = h_0$ and $\left.\frac{df}{dt}\right|_{t=0} = 0$.

Suppose that solutions exist of the form

$$f = f_0(t) + \varepsilon f_1(t) + \varepsilon^2 f_2(t) + \cdots$$

then, order by order, we get

$$\frac{d^2 f_0}{dt^2} = -f_0 \quad \text{with} \quad \begin{cases} f_0(0) = h_0 \\ \frac{df_0}{dt}(0) = 0 \end{cases}$$

$$\frac{d^2f_1}{dt^2} = -f_1 - f_0^3 \qquad \text{with} \quad \begin{cases} f_1(0) = 0 \\ \frac{df_1}{dt}(0) = 0 \end{cases}$$

$$\frac{d^2f_2}{dt^2} = -f_2 - 3f_0^2 f_1 \qquad \text{etc...}$$

The solution to the zeroth order with $f_0(0) = h_0$ is

$$f_0(t) = h_0 \cos t$$

Then we have

$$\frac{d^2f_1}{dt^2} = -f_1 - h_0^3(\cos t)^3 = -f_1 - h_0^3\left(\frac{1}{4}\cos 3t + \frac{3}{4}\cos t\right)$$

The general solution to the homogeneous problem is

$$f_1^G(t) = A\cos t + B\sin t$$

For the particular solution, the term in $\cos 3t$ simply suggest another term in $\cos 3t$. However, the one in $\cos t$ requires more care, & suggests a solution of the kind $t\sin t$.

$\to$ we try $f_1^{PS}(t) = K_1\cos 3t + K_2 t\sin t$.

then
$$\frac{df_1^{PS}}{dt} = -3K_1\sin 3t + K_2\sin t + K_2 t\cos t$$

$$\frac{d^2f_1^{PS}}{dt^2} = -9K_1\cos 3t + 2K_2\cos t - K_2 t\sin t$$

$$\Rightarrow \quad \frac{d^2f_1^{PS}}{dt^2} = -f_1^{PS} - \frac{h_0^3}{4}\cos 3t - \frac{3h_0^3}{4}\cos t$$

$$\Leftrightarrow \quad -9K_1\cos 3t + 2K_2\cos t - \cancel{K_2 t\sin t}$$
$$= -(K_1\cos 3t + \cancel{K_2 t\sin t}) - \frac{h_0^3}{4}\cos 3t - \frac{3h_0^3}{4}\cos t$$

$$\Rightarrow \quad -9K_1 = -K_1 - \frac{h_0^3}{4} \qquad 2K_2 = -\frac{3h_0^3}{4}$$

so $K_1 = \frac{h_o^3}{32}$ $\qquad K_2 = -\frac{3h_o^3}{8}$

and so $f_1(t) = A\cos t + B\sin t + \frac{h_o^3}{32}\cos 3t - \frac{3h_o^3}{8} t \sin t$

→ to satisfy the IC:

$$\begin{aligned} f_1(0) &= 0 \\ \frac{df_1}{dt} &= 0 \end{aligned} \quad \begin{cases} A + \frac{h_o^3}{32} = 0 \\ B = 0 \end{cases}$$

and finally, $f_1(t) = \frac{h_o^3}{32}(\cos 3t - \cos t) - \frac{3h_o^3}{8} t \sin t$

so

$$f(t) = h_o \cos t + \frac{\varepsilon h_o^3}{8}\left[\frac{\cos 3t}{4} - \frac{\cos t}{4} - 3t\sin t\right] + O(\varepsilon^2)$$

This expansion, however, is not uniform.
Suppose the first term is the one kept, and the second is the remainder, then we see that the latter cannot be bounded in a way that is independent of t because of the $t\sin t$ term.

This term is called a "secular" term. It becomes large as soon as t becomes of the order of $\frac{1}{\varepsilon}$
→ when that is the case, the remainder becomes of the same order as $h_o \cos t$.

In the next chapter, we will learn how to deal with secular terms.

Example 2 Small parameter multiplying highest derivative → singular limits.

In the previous chapter, we saw how small parameters multiplying the highest-order term in a polynomial lead to the emergence of singular solutions. Let's see what it does for ODEs.

Consider the very simple ODE $\varepsilon \frac{df}{dx} + f = e^{-x}$ where ε is small and positive, with $f(0) = 2$.

Assuming an expansion of the form

$$f = f_0(x) + \varepsilon f_1(x) + \varepsilon^2 f_2(x) \ldots$$

then, to lowest order and very naively, we would get

$$f_0(x) = e^{-x} \quad \text{with boundary condition } f_0(0) = 2$$

→ clearly that's impossible; Similarly to the next order we would just get $\frac{df_0}{dx} + f_1 = 0$

$$\Rightarrow \quad f_1 = -\frac{df_0}{dx} = e^{-x}$$

However, the BC to apply here would be $f_1(0) = 0$

→ again clearly impossible.

Let's now try to be smarter & remember that we should expect a singular expansion based on our discussion of the last chapter. In that case, let's assume

$$f(x) = \frac{1}{\varepsilon} f_{-1} + f_0 + \varepsilon f_1 + \ldots$$

then

$$\begin{cases} \text{To order } \frac{1}{\varepsilon} \Rightarrow & f_{-1} = 0 \quad \rightarrow \text{already problematic} \\ \text{To order } 1 \Rightarrow & \frac{df_{-1}}{dx} + f_0 = e^{-x} \\ \vdots \end{cases}$$

→ It looks like the solution found in the polynomial case does not work here. Why?

Since we can, actually, find the exact solution let's look at it to see what's going wrong.

Rewriting the equation as

$$\frac{df}{dx} = -\frac{1}{\varepsilon}f + \frac{1}{\varepsilon}e^{-x}$$ we see that

$f(x) = Ke^{-\frac{x}{\varepsilon}} + f^{PS}$, where

f^{PS} is of the form $f^{PS}(x) = Ae^{-x}$.

Plugging it in, we get

$$-A\cancel{e^{-x}} = -\frac{1}{\varepsilon}A\cancel{e^{-x}} + \frac{1}{\varepsilon}\cancel{e^{-x}}$$

$$\Rightarrow \left(\frac{1}{\varepsilon}-1\right)A = \frac{1}{\varepsilon} \quad \Rightarrow \quad A = \frac{1}{1-\varepsilon}$$

so $f(x) = Ke^{-x/\varepsilon} + \frac{1}{1-\varepsilon}e^{-x}$

Finally, to fit the IC, we get $2 = K + \frac{1}{1-\varepsilon}$ so

$$K = 2 - \frac{1}{1-\varepsilon} = \frac{2-2\varepsilon-1}{1-\varepsilon} = \frac{1-2\varepsilon}{1-\varepsilon}$$

So $f(x;\varepsilon) = \frac{1-2\varepsilon}{1-\varepsilon}e^{-\frac{x}{\varepsilon}} + \frac{1}{1-\varepsilon}e^{-x}$

Expanding this in powers of ε, we then get

$$f(x;\varepsilon) = (1-2\varepsilon)(1+\varepsilon+\varepsilon^2+\dots)e^{-\frac{x}{\varepsilon}} + (1+\varepsilon+\varepsilon^2+\dots)e^{-x}$$

↑ but what to do with this?

⇒ $e^{-x/\varepsilon}$ doesn't have an expansion in powers of ε.

→ It's no surprise our attempt to write f as an asymptotic sequence in ε failed..

However, we can still learn something from this.

For $x \gg \varepsilon$, the term in $e^{-x/\varepsilon}$ is exponentially small hence

$$f(x;\varepsilon) \simeq (1+\varepsilon+\varepsilon^2+\dots)e^{-x}$$

→ so the asymptotic sequence should work then. The key is <u>not</u> to worry about the initial conditions (yet) since the latter occur for $x \to 0$.

Going back to it, (assuming $f(x) = f_0(x)+\varepsilon f_1(x)+\dots$)

we set $\quad f_0(x) = e^{-x}$

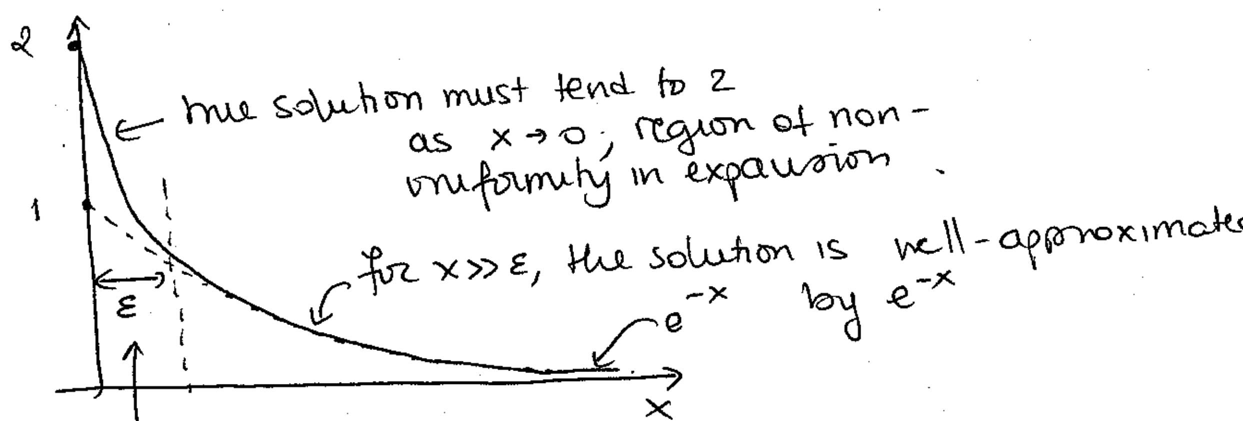
$$f_1(x) = e^{-x}$$

$$\frac{df_1}{dx} + f_2 = 0 \Rightarrow f_2 = -\frac{df_1}{dx} = e^{-x}$$

etc ... and indeed,

we then recover

$$f(x;\varepsilon) \simeq (1+\varepsilon+\varepsilon^2+\dots)e^{-x}$$ for large x (that is, $x \gg \varepsilon$)



but clearly, something else must be happening for $x \ll \varepsilon$ to match that solution to the ICs. This bit is captured (in the true solution) by the $e^{-x/\varepsilon}$ part, but can't be done with a simple asymptotic sequence

What happens as $x \to 0$ $(x < \varepsilon)$ is called a "boundary layer". The latter typically occur in ODES as the small parameter multiplies the highest order derivative.

Here, when $\varepsilon = 0$, the equation is <u>algebraic</u> so its not surprising we can't fit the solution to arbitrary boundary conditions. When $\varepsilon \neq 0$, however, we can – but that leads to the emergence of boundary layers, i.e., regions where the $\varepsilon \frac{df}{dx}$ term <u>is</u> important (even if it is multiplied by ε). This requires f to vary <u>rapidly</u> in the boundary layer.

In the next Chapters, we will learn to deal with boundary layers.