CHAPTER 2 Asymptotics

In this Chapter we now introduce some of the more formal definitions & tools that will be needed in this Perturbation methods course.

II.1 Order symbol O (big O)

- Recall the definition of limits from calculus:

If
$$\underbrace{\lim_{x \to x_{0-}} f(x)}_{\text{limit approaching } x_0 \text{ from the left}} = \underbrace{\lim_{x \to x_{0+}} f(x)}_{\text{limit approaching from right}} = f(x_0)$$

then we say that

① the limit $\lim_{x \to x_0} f(x)$ exists

② it is equal to $f(x_0)$

③ and the function $f(x)$ is continuous at $x = x_0$.

- Limits at ∞ can also be discussed in similar terms, by remembering how to perform a change of variables:

$$\lim_{x \to +\infty} f(x) = \lim_{y \to 0_+} f\left(\frac{1}{y}\right)$$

$$\lim_{x \to -\infty} f(x) = \lim_{y \to 0_-} f\left(\frac{1}{y}\right)$$

In this case, however, we only consider 1-sided limits.

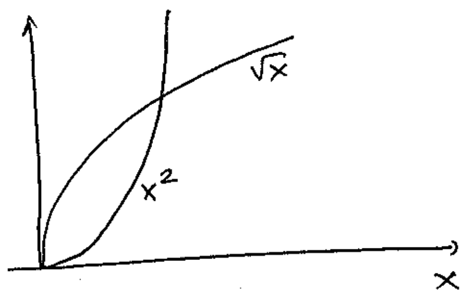
- Finally, note that we can <u>also</u> recast a limit at x_0 into a limit at 0 by another change of variables:

$$\lim_{x \to x_0} f(x) = \lim_{y \to 0} f(y + x_0)$$

In other words we can always treat a limit problem as one in which the independent variable tends to 0.

We now want to answer the question "How fast is $f(x)$ approaching its limit as $x \to 0_+$ or $x \to 0_-$?".

For instance, consider $f(x) = \sqrt{x}$ and $f(x) = x^2$

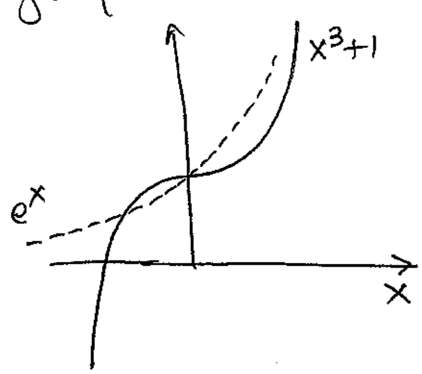


→ quite clear that x^2 goes to 0 much faster than $\sqrt{x}$

But how do we quantify / formalize this idea?

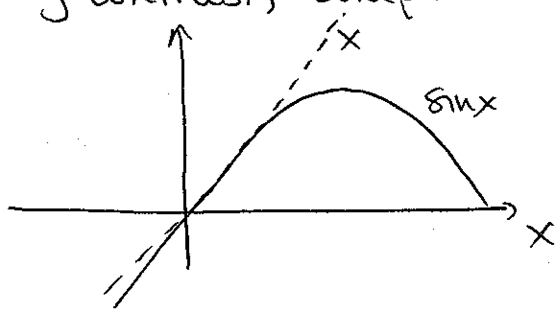
Similarly, consider $f(x) = x^3 + 1$ and $f(x) = e^x$

In both cases, $\lim_{x \to 0} f(x) = 1$. However, from the graphs we see that



→ here $x^3 + 1$ gets to 1 much faster than e^x does.

By contrast, compare $f(x) = x$ and $f(x) = \sin x$



→ $\sin x$ and x approach x at the same rate.

The concept of "how rapidly" something approaches 0 (or another limit) can be formalized with the order symbol O. (big O). We say that if $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$

and if $\lim_{x\to 0} \frac{f(x)}{g(x)} = C$

C a constant that is NOT 0

then $f(x) = O(g(x))$ as $x \to 0$.

and say "$f(x)$ is of the order of $g(x)$ as x tends to 0".

Example: $\lim_{x\to 0} \frac{1-\cos x}{x} = 0 \rightarrow$ $1-\cos x$ is NOT of the order of x

but $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2} \rightarrow$ $1-\cos x$ is of the order of x^2 as $x\to 0$.

Similarly we can do the same for limits at $x = x_0$ or limits at ∞: as long as $\lim f = \lim g$ at $x = x_0$ or $x \to \pm\infty$

Example: $\lim_{x\to +\infty} \frac{\frac{1}{x^2+2}}{\frac{1}{4x^2}} = 4$ so

$\frac{1}{x^2+2}$ is of the order of $\frac{1}{4x^2}$ as $x \to +\infty$.

$\lim_{x\to \frac{\pi}{2}} \frac{\sin(x-\frac{\pi}{2})}{x-\frac{\pi}{2}} = 1$ so

$\sin(x-\frac{\pi}{2})$ is of the order of $x-\frac{\pi}{2}$ as $x \to \frac{\pi}{2}$.

Note 1: In the expression $f(x) = O(g(x))$ $g(x)$ is called the gauge function. Usually $g(x)$ is taken to be a power of x; however, this is not always the case.

Note 2: It is now clear how to generalize this when the limits of f & g are not 0.

- if $\lim f = \lim g = L$ then we look at the limit of the ratio

$$\lim_{\substack{x \to x_0 \\ (\text{or } x \to \pm\infty)}} \frac{f(x)-L}{g(x)-L}$$

→ if it is a non-zero constant, then $f(x)-L$ is $O(g(x)-L)$ as $x \to x_0$ (or $\pm\infty$)

- if $\lim f$ & $\lim g$ are both $+\infty$ or $-\infty$, we can apply this directly, because it's the same as taking the ratio $\frac{1/f}{1/g}$ (where $1/f$ & $1/g$ both tend to 0)

Examples:

$x \to 0$ $\begin{cases} f(x) = x+1 \\ g(x) = \cos x \end{cases}$ $\Rightarrow$ $\lim\limits_{x \to 0} \dfrac{g(x)-1}{f(x)-1} = \lim\limits_{x \to 0} \dfrac{\cos x - 1}{x}$

$$= \lim_{x \to 0} -\frac{x}{2} = 0$$

→ $f(x)-1$ is not $O(g(x)-1)$ as $x \to 0$

$x \to +\infty$ $\begin{cases} f(x) = 4x^2 \\ g(x) = 6x^2+2 \end{cases}$

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = \frac{4x^2}{6x^2+2} = \frac{4}{6}$$

→ $f(x)$ is $O(g(x))$ as $x \to +\infty$

Note 3:- Since there is no power of x that goes to ∞ faster or as fast as e^x (or any exponential), we must allow for e^x to be a gauge function as well.

Similarly, since there is no power of x (negative power) that go to ∞ more slowly than $\log x$ as $x \to 0$, we also must allow for $\log x$ to be a gauge function.

→ In summary, we see that we will need to consider any power of x, any $\log(x)$, any e^x and any product thereof as gauge functions, at the very least.

II. 2 Taylor & McLaurin series with remainder

① Taylor series (infinite series)

Suppose that $f(x)$ is infinitely differentiable, then for x near enough x_0

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-x_0)^n}{n!} \left.\frac{d^n f}{dx^n}\right|_{x=x_0}$$

② Taylor's theorem for series with remainder

Suppose that $f(x)$ is differentiable at least N+1 times then for x near enough x_0

$$f(x) = \sum_{n=0}^{N} \frac{(x-x_0)^n}{n!} \left.\frac{d^n f}{dx^n}\right|_{x=x_0} + R_N$$

↑ remainder

where $R_N = \dfrac{(x-x_0)^{N+1}}{(N+1)!} \left.\dfrac{d^{N+1} f}{dx^{N+1}}\right|_{x=a}$

where a is a point lying between x and x_0.

Proof:

Note that $f(x) = f(x_0) + \int_{x_0}^{x} f'(t)\,dt$

then integrate the second term by parts;

$$f(x) = f(x_0) + \Big[(t-x)f'(t)\Big]_{x_0}^{x} + \int_{x_0}^{x} f''(t)(x-t)\,dt$$

IBP again

$$= f(x_0) + (x-x_0)f'(x_0) + \left[-\frac{(t-x)^2}{2}f''(t)\right]_{x_0}^{x} + \int_{x_0}^{x} f'''(t)\frac{(x-t)^2}{2}\,dt$$

$$= f(x_0) + (x-x_0)f'(x_0) + \frac{(x-x_0)^2}{2}f''(x_0) + \dots \text{ etc.}$$

after N iterations

$$= \sum_{n=0}^{N} \frac{(x-x_0)^n}{n!} f^{(n)}(x_0) + \int_{x_0}^{x} f^{(N+1)}(t)\frac{(x-t)^N}{N!}\,dt$$

Using mean value theorem + more calculus, we can then show that

$$\int_{x_0}^{x} f^{(N+1)}(t)\frac{(x-t)^N}{N!}\,dt = f^{(N+1)}(a)\frac{(x-x_0)^{N+1}}{(N+1)!}$$

where a is some point between x and x_0.

Notes: • If we consider $x_0 = 0$, then we get the McLaurin formula

$$f(x) = \sum_{n=0}^{N} \frac{x^n f^{(n)}(0)}{n!} + R_N \quad \text{where } R_N = \frac{x^{N+1}}{(N+1)!} f^{(N+1)}(a)$$

for $0 < a < x$

• Having assumed that the $N+1$ derivative exists and is continuous, we can then say that

$$\lim_{x \to x_0} \frac{R_N}{(x-x_0)^{N+1}} = \frac{f^{(N+1)}(a)}{(N+1)!} = \text{a constant}$$

$\rightarrow$ this then says that $R_N = O\left((x-x_0)^{N+1}\right)$

so we can write $f(x) = \displaystyle\sum_{n=0}^{N} \frac{(x-x_0)^n}{n!} \frac{d^n f}{dx^n}\Big|_{x_0} + O\left((x-x_0)^{N+1}\right)$

Examples of famous series

as $x \to 0$: $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

$$(1+x)^a = 1 + ax + \frac{a(a-1)}{2!} x^2 + \frac{a(a-1)(a-2)}{3!} x^3 + \cdots$$

(3) Two views of the Taylor Series:

Consider a function that is infinitely differentiable, and it's Taylor series with remainder near x_0

$$f(x) = \sum_{n=0}^{N} \frac{(x-x_0)^n}{n!} \frac{d^n f}{dx^n}\bigg|_{x_0} + \underbrace{\frac{(x-x_0)^{N+1}}{(N+1)!} \frac{d^{N+1} f}{dx^{N+1}}\bigg|_{x=a}}_{R_N}$$

Because we have shown that $R_N = O(x-x_0)^{N+1}$

we know that

$$\lim_{x \to x_0} f(x) - \sum_{n=0}^{N} \frac{(x-x_0)^n}{n!} \frac{d^n f}{dx^n}\bigg|_{x_0} = 0$$

$\to$ in otherwords, for fixed N,

the finite sum becomes a better & better approximation to $f(x)$ as $x \to x_0$

However, the "symmetric limit" for $N \to \infty$ at fixed x is much less obvious.

Indeed, if we consider a value of $x \neq x_0$ (but close to it

$$\lim_{N \to +\infty} \left[f(x) - \sum_{n=0}^{N} \frac{(x-x_0)^n}{n!} \frac{d^n f}{dx^n}\bigg|_{x_0} \right]$$

This limit is only equal to 0 iff

$$\lim_{N \to +\infty} \frac{(x-x_0)^{N+1}}{(N+1)!} \frac{d^{N+1} f}{dx^{N+1}}\bigg|_{x=a} = 0$$

→ This is a lot harder to prove, usually. Not all infinitely differentiable functions satisfy this.

<u>Definition</u>: if a function $f(x)$, infinitely differentiable, satisfies

$$\lim_{N \to \infty} \frac{(x-x_0)^N}{N!} \frac{d^N f}{dx^N}\bigg|_{x=a} = 0$$

for x in a <u>given</u> neighborhood D of x_0, and a is any point within D, then $f(x)$ is said to be <u>analytic</u> in D.

<u>Property of analytic functions</u>:

The Taylor series of an analytic function is convergent for any x in D

This means that for any x in D, we <u>can</u> get progressively <u>better</u> approximations to $f(x)$ by keeping more terms in the series.

This property, as well as its proof, is non-trivial.
- there are some functions for which this is never true (non-analytic functions)
- for x outside of D, the Taylor series does <u>not</u> converge as $N \to +\infty$ (meaning that the remainder does <u>NOT</u> tend to 0). Hence the qualifier, "for x close enough to x_0".

4) Convergent vs divergent series

To determine whether any series of the kind $\sum_{n=0}^{\infty} a_n$ converges, it suffices to show that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$.

This also then implies that $\lim_{n\to\infty} |a_n| = 0$. Hence, for a Taylor series to converge, we simply need to show that the remainders satisfy $\lim_{n\to\infty} \left|\frac{R_{n+1}}{R_n}\right| < 1$

Non-convergent series are not as big a problem as they may seem, however. Indeed, in perturbation methods we usually just keep a few terms in the series instead of seeking the limit $N \to \infty$. For instance, if we choose to only use the first 3 terms, we would write

$$f(x) \simeq f(x_0) + (x-x_0) f'(x_0) + \frac{(x-x_0)^2}{2} f''(x_0) + R_2$$

In that case, R_2 serves as an estimate of the error made in the approximation. While the remainder may not <u>always</u> tend to 0 as more terms are kept, it may still be small enough for the purposes of our approximate work.

See examples in pages 26–33 of textbook for more on this topic.

II.3 Little o symbol, and asymptotic sequences

① Little o

- Earlier we defined the O "big o" symbol to relate two functions satisfying:

if $\lim_{x \to x_0} \frac{f(x)}{g(x)} = C$ $(C \neq 0)$ then $f(x) = O(g(x))$ as $x \to x_0$.

- If, on the other hand

$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$ then $f(x) = o(g(x))$ as $x \to x_0$

little o.

This loosely means that $f(x)$ tends to zero much faster than $g(x)$ as $x \to x_0$.

Examples:

- $\lim_{x \to 0} \frac{\sin x}{x} = 1$ so $\sin x = O(x)$ as $x \to 0$

but $\lim_{x \to 0} \frac{\sin x}{\sqrt{x}} = 0$ so $\sin x = o(\sqrt{x})$ as $x \to 0$.

- $\lim_{x \to \infty} \frac{x}{e^x} = 0$ so $x = o(e^x)$ as $x \to +\infty$

and in fact, for any N, any $a > 1$

$\lim_{x \to \infty} \frac{x^N}{a^x} = 0$ so $x^N = o(a^x)$ as $x \to +\infty$.

Similarly

- $\lim_{x \to 0} x \ln x = 0$ as $x \to 0$ so $x = o\left(\frac{1}{\ln x}\right)$ as $x \to 0$

and in fact, for any N, and $a > 1$

$\lim_{x \to 0} x^N \log_a x = 0$ so $x^N = o\left(\frac{1}{\log_a x}\right)$ as $x \to 0$.

② Asymptotic sequences & series

Based on the definition of o, we can then define an asymptotic sequence as any combination of functions $\{\delta_0(\varepsilon), \delta_1(\varepsilon), \delta_2(\varepsilon), \delta_3(\varepsilon), \dots\}$ such that, $\forall n$, $\delta_{n+1}(\varepsilon) = o(\delta_n(\varepsilon))$

Example: • The sequence of functions

$$\{1, \varepsilon, \varepsilon^2, \varepsilon^3, \varepsilon^4, \dots\}$$

is an asymptotic sequence because $\varepsilon^{N+1} = o(\varepsilon^N)\ \forall N$.

• The sequence

$$\{1, \varepsilon^{1/2}, \varepsilon, \varepsilon^{3/2}, \dots\}$$

is an asymptotic sequence because $\varepsilon^{\frac{N+1}{2}} = o(\varepsilon^{\frac{N}{2}})\ \forall N$

Less obvious ones:

• The sequence

$$\{1, \ln(1+\varepsilon), \ln(1+\varepsilon^2), \ln(1+\varepsilon^3), \dots\}$$

is an asymptotic sequence.

To see this note that

$$\lim_{\varepsilon\to 0} \frac{\ln(1+\varepsilon^{N+1})}{\ln(1+\varepsilon^N)} = \lim_{\varepsilon\to 0} \frac{\varepsilon^{N+1}}{1+\varepsilon^{N+1}} \cdot \frac{1+\varepsilon^N}{\varepsilon^N}$$

(using l'Hopital's rule)

$$= 0 \quad \forall N.$$

Having defined an asymptotic sequence, note that a function $f(x)$ can, under some circumstances (see below) be written near x_0 as the asymptotic series

$$f(x) = \sum_{n=0}^{N-1} a_n \delta_n(x-x_0) + \underbrace{R_N}_{\text{remainder}}$$

• Note that the selection of the sequence itself is not unique, that is, we could choose many different sequences $\{\delta_n\}$ in the formula above. However, if an appropriate sequence has been selected, then the coefficients a_n are <u>unique</u> to that sequence.

<u>For instance:</u> If we want to approximate $\sin\varepsilon$ with the sequence $\{1, \varepsilon, \varepsilon^2, \varepsilon^3, \varepsilon^4, \varepsilon^5 \ldots\}$ we get the asymptotic series

$$\sin\varepsilon = \varepsilon - \frac{\varepsilon^3}{3!} + \frac{\varepsilon^5}{5!} \Rightarrow \begin{aligned} a_0 &= 0 \\ a_1 &= 1 \\ a_2 &= 0 \\ a_3 &= -\frac{1}{3!} \\ a_4 &= 0 \\ a_5 &= \frac{1}{5!} \end{aligned}$$

etc.

If, on the other hand, the selected sequence was $\{\varepsilon, \varepsilon^3, \varepsilon^5, \varepsilon^7 \ldots\}$ then

$$a_0 = 1, \quad a_1 = -\frac{1}{3!}, \quad a_2 = \frac{1}{5!}$$

etc...

This then shows that while the selection of an appropriate sequence for a function is not unique, not all sequences are appropriate either:

<u>Example:</u> $\{1, \varepsilon^2, \varepsilon^4, \varepsilon^6, \ldots\}$ is <u>NOT</u> a good sequence to use to represent $\sin\varepsilon$.

<u>Proof of uniqueness:</u> Suppose $f(\varepsilon) = \sum_{n=0}^{N} a_n \delta_n(\varepsilon) + R_N$

then $$f(\varepsilon) = a_0\delta_0(\varepsilon) + \sum_{n=1}^{N} a_n \delta_n(\varepsilon) + R_N$$

so $$\lim_{\varepsilon\to 0} \frac{f(\varepsilon)}{\delta_0(\varepsilon)} = a_0 + \sum_{n=1}^{N} a_n \lim_{\varepsilon\to 0} \frac{\delta_n(\varepsilon)}{\delta_0(\varepsilon)} + \lim_{\varepsilon\to 0} \frac{R_N}{\delta_0(\varepsilon)}$$

$$= a_0 + 0 + 0 \leftarrow \text{by definition}$$

Hence a_0 is uniquely determined to be

$$a_0 = \lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{\delta_0(\varepsilon)}.$$

Similarly, it's easy to show that

$$a_1 = \lim_{\varepsilon \to 0} \frac{f(\varepsilon) - a_0\delta_0(\varepsilon)}{\delta_1(\varepsilon)}$$

$$\vdots$$

$$a_n = \lim_{\varepsilon \to 0} \frac{f(\varepsilon) - \sum_{k=0}^{n-1} a_k \delta_k(\varepsilon)}{\delta_n(\varepsilon)}$$

③ Uniform & non-uniform expansions

Suppose now that the function $f(x)$ is solution of an equation containing the small parameter ε (as in the first example of Chapter 1). Then the solution, ideally, should be expanded as

$$f(x;\varepsilon) = \sum_{n=0}^{N} a_n(x)\,\delta_n(\varepsilon) + R_N(x;\varepsilon)$$

where $R_N(x,\varepsilon) = O(\delta_{n+1}(\varepsilon))$

where $\{\delta_n(\varepsilon)\}$ is an asymptotic sequence for small ε. Here the coefficients of the series representing f are themselves functions of the independent variable.

Examples

- In chapter 1, we had

$$w(t;\varepsilon) = -t + \varepsilon\frac{t^2}{2} - \varepsilon^2\frac{t^3}{6} + \cdots$$

- Suppose the exact solution of an equation is $f(x;\varepsilon) = \dfrac{1}{1-\varepsilon\sin x}$

Then $f(x;\varepsilon) = 1 + \varepsilon\sin x + \varepsilon^2\sin^2 x + \varepsilon^3 \sin^3 x + \cdots$

The behaviour of these two series is very different, however.

In the first case, for large t, we see that the coefficients in front of $\delta_n(\varepsilon)$ increase with n. In other words, for larger values of t one must go to much larger N before convergence is possible

$$\left|\frac{R_{N+1}}{R_N}\right| = \frac{(N+1)!}{(N+2)!}\frac{\varepsilon^{N+1}}{\varepsilon^N}\frac{t^{N+2}}{t^{N+1}} = \frac{\varepsilon t}{N+2}$$ While $\lim_{N\to+\infty}\left|\frac{R_{N+1}}{R_N}\right| < 1$

at given ε, given t, one has to go to much larger N before $\left|\frac{R_{N+1}}{R_N}\right| < 1$ if t is large.

$\Rightarrow$ NONUNIFORM CONVERGENCE WITH t.

Even more extreme examples exist where the series converges for some values of t, but not others.

<u>Definition</u>: A <u>uniform</u> asymptotic expansion is one in which the remainder $R_N(x,\varepsilon)$ satisfies $|R_N(x;\varepsilon)| \leq K\,\delta_{N+1}(\varepsilon)$

$\uparrow$ a constant <u>independent</u> of x.

In this example we have $R_N(x;\varepsilon) \sim K t^{N+1}\varepsilon^N$

$\rightarrow$ this <u>cannot</u> be bounded independently of t.

However, in the second example

$$f(x;\varepsilon) = 1 + \varepsilon\sin x + \varepsilon^2\sin^2 x + \cdots$$

then $R_N \sim \varepsilon^3\sin^3 x$

$\rightarrow |R_N| \sim \varepsilon^3|\sin^3 x| \leq \varepsilon^3$

$\uparrow$ <u>CAN</u> be bounded