I. 3 Approximations of functions

So far we have learned how to approximate solutions of ODEs, and solutions of polynomials, i.e quantities defined by a complex equation.

In fact it often happens that we may want to do something much simpler involving evaluating a function at a point $x=a$, or near a point $x=a$.

Example 1: suppose we want to evaluate $e^{0.1}$ but we can't get hold of a calculator.

→ Recall the expansion of e^x near $x=a=0$

$$e^x = 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots \quad \text{for small } x.$$

$$\rightarrow e^{0.1} = 1+0.1+\frac{0.01}{2}+\frac{0.001}{6}+\cdots$$

$$= 1+0.1+0.005+0.00016\ldots \cong 1.105166\ldots$$

(The exact value is $1.105171\ldots$)

Similar calculations can be used to obtain approximate values of $f(x)$ near a point $a \neq 0$:

Example 2 What is $\sqrt{37}$ (approximately?) without using $\sqrt{\ }$ function?

Since $\sqrt{37} = \sqrt{36+1}$

and $\sqrt{a+\epsilon} = \sqrt{a}\left(1+\frac{\epsilon}{a}\right)^{1/2} \cong \sqrt{a}\left(1+\frac{1}{2}\frac{\epsilon}{a}+\frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{2}\frac{\epsilon^2}{a^2}+\cdots\right)$

then $\sqrt{37} = \sqrt{36}\left(1+\frac{1}{2}\frac{1}{36}-\frac{1}{8}\left(\frac{1}{36}\right)^2+\cdots\right)$

$$= 6\left(1+\frac{1}{72}-\frac{1}{8(36)^2}\cdots\right) \simeq 6.082\ldots \quad (\text{as in } \sqrt{37})$$

Example 3 Taylor expansions can also be used to approximate functions defined by integrals, such as the error function:

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

$$\rightarrow \quad \mathrm{erf}(\epsilon) = \mathrm{erf}(0) + \epsilon \left.\frac{d.\,\mathrm{erf}}{dx}\right|_0 + \frac{\epsilon^2}{2} \left.\frac{d^2\mathrm{erf}}{dx^2}\right|_0 + \frac{\epsilon^3}{6}\frac{d^3\mathrm{erf}}{dx^3} + \ldots$$

$$= 0 + \epsilon \cdot \frac{2}{\sqrt{\pi}}\left[e^{-t^2}\right]_0 + \frac{\epsilon^2}{2}\cdot\frac{2}{\sqrt{\pi}}\left[-2te^{-t^2}\right]_0$$

$$+ \frac{\epsilon^3}{6}\frac{2}{\sqrt{\pi}}\left[-2e^{-t^2} + 4t^2e^{-t^2}\right]_0 + \ldots$$

$$= \frac{2}{\sqrt{\pi}}\left(\epsilon - \frac{\epsilon^3}{3} + \ldots\right)$$

This method, however, rapidly becomes quite painful. Note, however, since ϵ is small, any t in the interval $[0, \epsilon]$ is <u>also</u> small so we can expand the integrand instead:

$$\rightarrow \quad \mathrm{erf}(\epsilon) = \frac{2}{\sqrt{\pi}}\int_0^\epsilon e^{-t^2}\, dt \simeq \frac{2}{\sqrt{\pi}}\int_0^\epsilon \left(1 - t^2 + \frac{1}{2}t^4 - \frac{1}{6}t^6 + \ldots\right) dt$$

$$= \frac{2}{\sqrt{\pi}}\left(\epsilon - \frac{\epsilon^3}{3} + \frac{\epsilon^5}{10} - \frac{\epsilon^7}{42} + \ldots\right)$$

much more efficient!

Example 4 In all cases above, we were interested in expansions near $x=a$ where a is finite. However it is also possible to get an expansion of a function near ∞, simply by letting $\epsilon = \frac{1}{x}$ and rewriting the function appropriately.

- For instance, we know that $\lim_{x\to+\infty} \frac{1+x}{1-2x} = -\frac{1}{2}$ but how does this function approach ∞?

If $x = \frac{1}{\epsilon}$ then $f\left(\frac{1}{\epsilon}\right) = \dfrac{1+\frac{1}{\epsilon}}{1-\frac{2}{\epsilon}} = \dfrac{\epsilon+1}{\epsilon-2} = g(\epsilon)$

If ϵ is small then

$$g(\epsilon) = \frac{(\epsilon+1)}{-2}\frac{1}{\left(1-\frac{\epsilon}{2}\right)} \simeq \frac{\epsilon+1}{-2}\left(1+\frac{\epsilon}{2}+\frac{\epsilon^2}{4}+\dots\right)$$

$$\simeq -\frac{1}{2} + \epsilon\left(-\frac{1}{2}-\frac{1}{4}\right) + \epsilon^2\left(-\frac{1}{4}-\frac{1}{8}\right) + \dots$$

$$\simeq -\frac{1}{2} - \frac{3\epsilon}{4} - \frac{3\epsilon^2}{8} + \dots$$

which implies in return that

$$f(x) \simeq -\frac{1}{2} - \frac{3}{4x} - \frac{3}{8x^2} \quad \text{when } x \to +\infty.$$

- We can try to do the same for the error function to get an estimate of its behaviour as $x \to +\infty$

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt = \frac{2}{\sqrt{\pi}}\left[\int_0^\infty e^{-t^2}\,dt - \int_x^\infty e^{-t^2}\,dt\right].$$

$$= \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}\,dt - \mathrm{erfc}(x)$$

complementary erf.

Unfortunately, erfc(x) doesn't lend itself to simple expansions as $x \to +\infty$.

Eg. If we try $g(\epsilon) = \text{erfc}\left(\frac{1}{\epsilon}\right) = \frac{2}{\sqrt{\pi}} \int_{\frac{1}{\epsilon}}^{\infty} e^{-t^2}\, dt$

Change $u = \frac{1}{t}$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\epsilon} e^{-\frac{1}{u^2}} \frac{du}{u^2}$$

This term does not have a regular taylor expansion near 0.

On the other hand here we can use a trick:

$$\frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2}\, dt = \frac{2}{\sqrt{\pi}} \int_x^{\infty} \frac{-2t\, e^{-t^2}}{-2t}\, dt \overset{\text{IBP}}{=} \frac{2}{\sqrt{\pi}} \left\{ \left[\frac{e^{-t^2}}{-2t} \right]_x^{\infty} - \int_x^{\infty} \frac{e^{-t^2}}{2t^2}\, dt \right\}$$

$$= \frac{2}{\sqrt{\pi}} \left\{ \frac{e^{-x^2}}{2x} - \frac{1}{2} \int_x^{\infty} \frac{e^{-t^2}}{t^2}\, dt \right\}$$

use same trick

$$= \frac{2}{\sqrt{\pi}} \left\{ \frac{e^{-x^2}}{2x} + \frac{1}{2} \int_x^{\infty} \frac{-2te^{-t^2}}{2t^3}\, dt \right\}$$

IBP

$$= \frac{2}{\sqrt{\pi}} \left\{ \frac{e^{-x^2}}{2x} + \frac{1}{2} \left[\frac{e^{-t^2}}{2t^3} \right]_x^{\infty} + \frac{3}{4} \int_x^{\infty} \frac{e^{-t^2}}{t^4}\, dt \right\}$$

$$= \frac{2}{\sqrt{\pi}} e^{-x^2} \left\{ \frac{1}{2x} - \frac{1}{4x^3} + \cdots \right\}$$

So finally, near $x = \infty$,

$$\text{erf}(x) \simeq 1 - \frac{e^{-x^2}}{\sqrt{\pi}} \left\{ \frac{1}{x} - \frac{1}{2x^3} + \cdots \right\}.$$

To summarize, in this section we saw that there are many ways to approximate functions at a point or near ∞, some of which are obvious (using Taylor expansions) and some of which require smart tricks. Throughout the course, we will see more of these tricks, in particular when trying to approximate functions defined as integrals (as in the case of the error function).

However, in all that preceeded, we never asked the question of whether the series of terms derived to approximate a function actually converges. This, in fact, is not always the case. However, there are tools to determine convergence and even in the absence of convergence, other tools to bound the error made in approximating the function.

This, and other more formal mathematical methods that form the basis of perturbation theory, are now introduced in Chapter 2.