In what follows, we will attempt to find asymptotic solutions of $\frac{d^2f}{dx^2} + p(x)\frac{df}{dx} + q(x)f = 0$ as $x \to \pm\infty$ in the case where $x \to \pm\infty$ is a singular (irregular) point – in which case the Frobenius expansion doesn't work.

An important tool for solving the issue lies in the method of dominant balance, which we now re-visit more formally

II The method of dominant balance

<u>Recall</u>: The idea of this method is to find a balance between two dominant terms, and deduce from that balance what the behavior of the solution may be.

To formalize it, consider an equation – either an algebraic equation for x, or an ODE for $f(x)$, or even a PDE, etc... ; this equation contains a small parameter ϵ. The method of dominant balance begins by assuming that the solution can be expressed as an asymptotic series, e.g

$$x = x_0(\epsilon) + x_1(\epsilon) + x_2(\epsilon) + \cdots$$ where $x_j(\epsilon) = o(x_i(\epsilon))$

or $$f(x) = f_0(x;\epsilon) + f_1(x;\epsilon) + \cdots$$ where $f_j = o(f_i)$ when $j > i$

⋮

and then seeks <u>some</u> plausible solutions.

Example 1 $\quad x^4 + 3x^3 + 2x^2 - \varepsilon = 0$

let $\quad x = x_0(\varepsilon) + x_1(\varepsilon) + \cdots$

→ To the lowest order, we have

$$x_0^4 + 3x_0^3 + 2x_0^2 - \varepsilon = 0$$

(we have to keep the ε term since we dont yet know what order x_0 is).

let's now see what terms may balance out:

- Either x_0 is $O(1)$ in which case

$$x_0^4 + 3x_0^3 + 2x_0^2 = 0$$

$$\Rightarrow \quad x_0^2 (x_0^2 + 3x_0 + 2) = 0$$

$$\Rightarrow \quad x_0^2 (x_0+2)(x_0+1) = 0 \rightarrow \begin{cases} x_0 = 0 \\ x_0 = -1 \\ x_0 = -2 \end{cases}$$

The first solution is not $O(1)$, but the others are. let's keep them in mind as the first two "seeds" of possible solutions

- If x_0 is not $O(1)$, then the term in ε is important. In that case the balance could be between

$$\begin{cases} x_0^4 - \varepsilon = 0 & \rightarrow \quad x_0 = O(\varepsilon^{1/4}) \\ 3x_0^3 - \varepsilon = 0 & \rightarrow \quad x_0 = O(\varepsilon^{1/3}) \\ 2x_0^2 - \varepsilon = 0 & \rightarrow \quad x_0 = O(\varepsilon^{1/2}) \end{cases}$$

let's see, by calculating the other terms, whether any of this could work

- If $x_0 = O(\varepsilon^{1/4})$ then, say $x_0^2 = O(\varepsilon^{1/2})$ which is larger than ε so we shouldn't have neglected it → contradiction

- The same applies to the case $x_0 = O(\varepsilon^{1/3})$
- For $x_0 = O(\varepsilon^{1/2})$ then $x_0^3 = O(\varepsilon^{3/2})$ and $x_0^4 = O(\varepsilon^2)$ so the terms $x_0^4 + 3x_0^3$ are indeed negligible

→ another set of 2 "seeds" is given by

$$2x_0^2 - \varepsilon = 0 \quad \Rightarrow \quad x_0(\varepsilon) = \pm\sqrt{\frac{\varepsilon}{2}}$$

This gives us for possible seeds for the lowest order of the asymptotic series: $x_0(\varepsilon) = -1$, $x_0(\varepsilon) = -2$, $x_0(\varepsilon) = \pm\sqrt{\frac{\varepsilon}{2}}$

→ <u>To the next order</u> (how to get $x_1(\varepsilon)$)

Consider for instance the seed $x_0(\varepsilon) = -1$, and let

$$x = -1 + x_1(\varepsilon) + \dots \text{ with the assumption } x_1(\varepsilon) = o(1)$$

Then $(-1 + x_1(\varepsilon) + \dots)^4 + 3(-1 + x_1(\varepsilon) + \dots)^3 + 2(-1 + x_1(\varepsilon) + \dots)^2 - \varepsilon = 0$

$$\Rightarrow \quad 1 - 4x_1(\varepsilon) + \dots + 3(-1 + 3x_1(\varepsilon) + \dots) + 2(1 - 2x_1(\varepsilon) + \dots) - \varepsilon = 0$$

$$\Rightarrow \quad -4x_1(\varepsilon) + 9x_1(\varepsilon) - 4x_1(\varepsilon) - \varepsilon = 0$$

$$\Rightarrow \quad x_1(\varepsilon) = \varepsilon$$

Note that this time we can neglect terms in x_1^2, x_1^3 etc since we know $x_1 = o(1)$

In this case, finding $x_1(\varepsilon)$ is easy! The next term for the seed $x_0(\varepsilon) = -2$ is also easy to find.

Now consider the seed $x_0(\varepsilon) = \sqrt{\frac{\varepsilon}{2}}$. Then let

$$x = \sqrt{\frac{\varepsilon}{2}} + x_1(\varepsilon) + \dots \quad \text{where } x_1(\varepsilon) = o(\varepsilon^{1/2})$$

$$\Rightarrow \quad \left(\sqrt{\frac{\varepsilon}{2}} + x_1(\varepsilon) + \dots\right)^4 + 3\left(\sqrt{\frac{\varepsilon}{2}} + x_1(\varepsilon) + \dots\right)^3 + 2\left(\sqrt{\frac{\varepsilon}{2}} + x_1(\varepsilon) + \dots\right)^2 - \varepsilon = 0$$

$$\Rightarrow \quad \frac{\varepsilon^2}{4}\left(1 + \sqrt{\frac{2}{\varepsilon}}\,x_1(\varepsilon) + \dots\right)^4 + \frac{3\varepsilon^{3/2}}{2^{3/2}}\left(1 + \sqrt{\frac{2}{\varepsilon}}\,x_1(\varepsilon) + \dots\right)^3 + \varepsilon\left(1 + \sqrt{\frac{2}{\varepsilon}}\,x_1(\varepsilon) + \dots\right)^2 - \varepsilon = 0$$

since $x_1(\varepsilon) = o(\sqrt{\varepsilon})$, $\sqrt{\frac{2}{\varepsilon}}\,x_1(\varepsilon) = o(1)$

so again we can neglect h.o.t in powers of $\sqrt{\frac{2}{\varepsilon}}x_1$.

$$\Rightarrow \quad \frac{\varepsilon^2}{4}\left(1+4\sqrt{\frac{2}{\varepsilon}}x_1+\cdots\right)+3\left(\frac{\varepsilon}{2}\right)^{3/2}\left(1+3\sqrt{\frac{2}{\varepsilon}}x_1+\cdots\right)+2\sqrt{2\varepsilon}\,x_1(\varepsilon)=0$$

Keeping only the identifiably-lowest orders, we get:

$$\underbrace{\cancel{\frac{\varepsilon^2}{4}}}_{\text{neglect}}+3\left(\frac{\varepsilon}{2}\right)^{3/2}+x_1(\varepsilon)\Big[\underbrace{\cancel{\sqrt{2}\,\varepsilon^{3/2}}}_{\text{neglect}}+\underbrace{\cancel{\tfrac{9}{2}\varepsilon}}_{\text{neglect}}+2\sqrt{2\varepsilon}\Big]=0$$

$$\Rightarrow \quad 3\left(\frac{\varepsilon}{2}\right)^{3/2}+x_1\cdot 2\sqrt{2\varepsilon}=0$$

$$\Rightarrow \quad x_1(\varepsilon)=-\frac{3\varepsilon}{8}$$

In that case we see that provided we keep good track of how small a term is, it's also easy to find x_1. (albeit not quite as above)

<u>Example 2</u> Consider $\frac{d^2f}{dx^2}+f=\frac{1}{x}$ and seek solutions in the limit $x\to\pm\infty$.

- This being a linear non-homogeneous equation we know that solutions must be of the kind
$$f(x)=A\cos x+B\sin x+f_{PS}(x)$$
where $f_{PS}(x)$ solves $\frac{d^2f_{PS}}{dx^2}+f_{PS}=\frac{1}{x}$

- Since $f_{PS}(x)$ <u>must</u> know about the RHS, the dominant balance must either be between $\frac{d^2f_{PS}}{dx^2}=\frac{1}{x}$ or $f_{PS}=\frac{1}{x}$

Let's formalize this. Let $f_{PS}=f_0(x)+f_1(x)+\cdots$ where $f_1(x)=o(f_0(x))$ as $x\to\pm\infty$

To lowest order, we have $\frac{d^2f_0}{dx^2}+f_0=\frac{1}{x}$

<u>Case 1</u>: $\frac{d^2f_0}{dx^2}=\frac{1}{x} \Rightarrow \frac{df_0}{dx}=\ln|x|+C$

$$\Rightarrow f_0(x)=x\ln|x|-x+Cx+D.$$

These all go to $\pm\infty$ as x goes to $\pm\infty$ $\rightarrow$

f_0 is <u>not</u> negligible $\rightarrow$ contradiction.

<u>Case 2:</u> $f_0 = \frac{1}{x}$ then $\frac{df_0}{dx} = -\frac{1}{x^2}$ then $\frac{d^2 f_0}{dx^2} = +\frac{2}{x^3}$

$$\frac{2}{x^3} = o\left(\frac{1}{x}\right) \text{ as } x \rightarrow \pm\infty \text{ so that works!}$$

Let's now find $f_1(x)$ from this seed.

Let $f(x) = \frac{1}{x} + f_1(x) \Rightarrow$

$$\frac{2}{x^3} + f_1''(x) + \frac{1}{x} + f_1(x) = \frac{1}{x} \Rightarrow$$

$$f_1''(x) + f_1(x) = -\frac{2}{x^3}$$

By analogy with earlier, we see that as $x \rightarrow \pm\infty$,
$f_1(x) \simeq -\frac{2}{x^3}$ (then $f_1''(x) = o(f_1(x))$ and can be neglected)

$$\rightarrow \quad f_{PS}(x) = \frac{1}{x} - \frac{2}{x^3} + \cdots$$

$\Rightarrow$ our solution, as $x \rightarrow \pm\infty$, can be written as

$$\boxed{f(x) = A\cos x + B\sin x + \frac{1}{x} - \frac{2}{x^3} + \cdots}$$

Note how this example was made particularly easy by the fact that we had a term $\left(\frac{1}{x}\right)$ whose behavior we knew as $x \rightarrow \pm\infty$. Compare instead the problems where

$$\frac{d^2 f}{dx^2} + f = -f^3 \quad \text{or} \quad \frac{d^2 f}{dx^2} = \frac{f}{x}$$

This is harder because we don't have a point of comparison. See later for a solution to the problem

III Behavior near $x=\infty$ of second-order linear ODEs

We now consider the general expression

$$\frac{d^2f}{dx^2} + p(x)\frac{df}{dx} + q(x)f = 0$$

& wish to find an approximation to $f(x)$ as $x \to \infty$.

If this equation has a regular or regular-singular point at ∞, then we could simply use a Frobenius series. However, let's now assume that $\lim_{x\to\infty} x^2 q(x)$ does <u>NOT</u> exist.

① Removal of the first derivative

To simplify the following calculation, first note that it is always possible to transform

$$\frac{d^2f}{dx^2} + p(x)\frac{df}{dx} + q(x)f = 0 \quad \text{into}$$

$$\frac{d^2y}{dx^2} + Q(x)y = 0 \quad \text{by suitable}$$

manipulations.

→ Use an integrating factor!

Indeed let $y(x) = f(x)e^{\frac{1}{2}\int p(x)dx}$

Then $\dfrac{df}{dx} = \dfrac{d}{dx}\left[y(x)e^{-\frac{1}{2}\int p(x)dx}\right]$

$$= \frac{dy}{dx}e^{-\frac{1}{2}\int p(x)dx} - \frac{p(x)}{2}y(x)e^{-\frac{1}{2}\int p(x)dx}$$

$$\frac{d^2f}{dx^2} = \frac{d^2y}{dx^2}e^{-\frac{1}{2}\int p\,dx} - p(x)\frac{dy}{dx}e^{-\frac{1}{2}\int p\,dx}$$

$$- \frac{y(x)}{2}p'(x)e^{-\frac{1}{2}\int p\,dx} + \frac{1}{4}p^2 y e^{-\frac{1}{2}\int}$$

$\Rightarrow$ the original equation becomes (dividing by $e^{-\frac{1}{2}\int p\,dx}$)

$$\frac{d^2y}{dx^2} - \cancel{p(x)\frac{dy}{dx}} - \frac{y(x)}{2}p'(x) + \frac{1}{4}p^2(x)y(x)$$
$$+\left[\cancel{\frac{dy}{dx}} - \frac{p(x)}{2}y(x)\right]p(x) + q(x)y(x) = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} + y(x)\underbrace{\left[q(x) - \frac{p'(x)}{2} - \frac{1}{4}p^2(x)\right]}_{Q(x)} = 0$$

② Asymptotic expansion for $y(x)$ as $x \to \pm\infty$

We now learn of a method to find the general behavior of $\frac{d^2y}{dx^2} + Q(x)y = 0$ as $x \to \pm\infty$

As we saw before, it's a little harder to construct an asymptotic series because we don't have a point of comparison for y. $\rightarrow$ use a trick: instead

of $y(x) = y_0 + y_1 + \cdots$ where y_n are asymptotic sequence

we let $y(x) = e^{\phi_0(x) + \phi_1(x) + \cdots} = e^{\sum_{n=0}^{\infty}\phi_n(x)}$ where ϕ_n are an asymptotic sequence

What does this buy us?

well $$\frac{dy}{dx} = \left(\sum_{n=0}^{\infty}\phi_n'(x)\right)e^{\sum_{n=0}^{\infty}\phi_n(x)}$$

$$\frac{d^2y}{dx^2} = \left(\sum_{n=0}^{\infty}\phi_n''(x)\right)e^{\sum_{n=0}^{\infty}\phi_n(x)} + \left(\sum_{n=0}^{\infty}\phi_n'(x)\right)^2 e^{\sum_{n=0}^{\infty}\phi_n(x)}$$

$\Rightarrow \frac{d^2y}{dx^2} + Q(x)y = 0$ becomes (dividing by $e^{\sum_{n=0}^{\infty}\phi_n}$)

$$\sum_{n=0}^{\infty}\phi_n''(x) + \left(\sum_{n=0}^{\infty}\phi_n'(x)\right)^2 + Q(x) = 0$$

↑ now independent of y

Since the Φ_ns form an asymptotic sequence, we can then use the method of dominant balance to solve for them one by one!

Example 1 : Airy's equation: $\frac{d^2y}{dx^2} = xy$

$\rightarrow$ already in form without 1st derivative

So let $y(x) = e^{\sum_{n=0}^{\infty}\phi_n(x)}$

$$\Rightarrow \sum_{n=0}^{\infty} \phi_n''(x) + \left(\sum_{n=0}^{\infty} \phi_n'(x)\right)^2 - x = 0.$$

To lowest order, we have:

$$\phi_0'' + \phi_0'^2 - x = 0$$

$\rightarrow$ three possibilities:

- $\phi_0'' = x \Rightarrow \phi_0' = \frac{x^2}{2} + C \qquad \phi_0 = \frac{x^3}{6} + Cx + D.$

  but then $\phi_0'^2 \sim \frac{x^4}{4} + \cdots \rightarrow \gg \phi_0''$

  $\rightarrow$ inconsistent.

- $\phi_0'^2 = x$ $\begin{cases} \phi_0' = \pm\sqrt{x} \text{ if } x > 0 \Rightarrow \phi_0(x) = \pm \frac{2}{3} x^{3/2} + C \\ \phi_0' = \pm i\sqrt{x} \text{ if } x < 0 \Rightarrow \phi_0(x) = \pm i \frac{2}{3}|x|^{3/2} + C \end{cases}$

  In both cases, $\phi_0'' = O\left(\frac{1}{\sqrt{x}}\right) \rightarrow$ could work

- $\phi_0'' + \phi_0'^2 = 0.$

  $\rightarrow$ let's define $\varphi = \phi_0' \Rightarrow \varphi' + \varphi^2 = 0$

  $$\frac{d\varphi}{\varphi^2} = -dx \qquad -\frac{1}{\varphi} = -x + C$$

  $$\rightarrow \varphi = \frac{1}{x-C} = \phi_0'$$

  $$\rightarrow \phi_0'^2(x) = \left(\frac{1}{x-C}\right)^2$$

  this is $\ll x$ for large x so inconsistent

→ our correct seeds correspond to $\phi_0(x) = o\left(\frac{2}{3}x^{3/2}\right)$

We now have to continue with the cases $x>0$ and $x<0$ separately. Let's start with $x>0$ ($x \to +\infty$).

Keeping terms up to the next order in the expansion we have

$$\phi_0''(x) + \phi_1''(x) \dots + (\phi_0' + \phi_1' + \dots)^2 - x = 0$$

$$\Rightarrow \quad \phi_0''(x) + \phi_1''(x) + \dots + \phi_0'^2\left(1 + \frac{\phi_1'}{\phi_0'} + \dots\right)^2 - x = 0$$

this is $o(1)$ so neglect any higher powers of this term

$$\Rightarrow \quad \phi_0''(x) + \phi_1''(x) + \dots + \cancel{\phi_0'^2} + \phi_0'^2 \cdot 2\frac{\phi_1'}{\phi_0'} - \cancel{x} = 0$$

$$\Rightarrow \quad \phi_0'' + \phi_1''(x) + 2\phi_0'\phi_1' = 0$$

$$\Rightarrow \quad \phi_1'' \pm 2\sqrt{x}\,\phi_1' = \mp\frac{1}{2\sqrt{x}}$$

$$\Rightarrow \quad \text{let } \varphi = \phi_1' \quad \Rightarrow \quad \varphi' \pm 2\sqrt{x}\,\varphi = \mp\frac{1}{2\sqrt{x}}$$

We could try to solve this using IF method, but that turns out to be quite hard. However, we can use dominant balance here too:

- $\varphi' = \mp\frac{1}{2\sqrt{x}} \Rightarrow \varphi = C \mp 2\sqrt{x}$ but then

  $\pm 2\sqrt{x} \cdot (C \mp 2\sqrt{x}) = o(x)$

  which is NOT negligible compared with other terms → inconsistent

- $\varphi' \pm 2\sqrt{x}\,\varphi = 0 \Rightarrow \frac{d\varphi}{\varphi} = \mp 2\sqrt{x}\,dx$

  $$\ln\varphi = \mp\frac{4}{3}x^{3/2} + C$$

  $$\varphi = K e^{\mp\frac{4}{3}x^{3/2}}$$

The term in $e^{+\frac{4}{3}x^{3/2}}$ is not small, but that in $e^{-\frac{4}{3}x^{3/2}}$ is. → could work ..

$$\therefore \pm 2\sqrt{x}\,\varphi = \mp \frac{1}{2\sqrt{x}} \Rightarrow \varphi = -\frac{1}{4x}$$

$$\Rightarrow \frac{d\varphi}{dx} = +\frac{1}{4x^2}$$ → is always small compared with $2\sqrt{x}\,\varphi$.

So for $\phi_1(x)$, we either get

$$\phi_1(x) = -\frac{1}{4}\ln x + K, \quad \underbrace{\phi_1(x) = \int K e^{-\frac{4}{3}x^{3/2}}\,dx}$$

this term is exponentially small as $x \to +\infty$ → probably negligible (see later)

To find $\phi_2(x)$, we repeat again...

$$\phi_0'' + \phi_1'' + \phi_2'' \ldots + (\phi_0' + \phi_1' + \phi_2' + \ldots)^2 - x = 0$$

$$\hookrightarrow \phi_0'' + \phi_1'' + \phi_2'' + \phi_0'^2\left(1 + \frac{\phi_1'}{\phi_0'} + \frac{\phi_2'}{\phi_0'} + \ldots\right)^2 - x = 0$$

$$\hookrightarrow \cancel{\phi_0''} + \phi_1'' + \phi_2'' + \phi_0'^2\left(\cancel{1} + \cancel{\frac{2\phi_1'}{\phi_0'}} + \frac{2\phi_2'}{\phi_0'} + \frac{\phi_1'^2}{\phi_0'^2} + \ldots\right) \cancel{- x} = 0$$

$$\hookrightarrow \phi_1'' + \phi_2'' + 2\phi_2'\phi_0' + \phi_1'^2 = 0.$$

$$\hookrightarrow \phi_2'' \pm 2\sqrt{x}\,\phi_2' = -\frac{1}{4x^2} - \frac{1}{16x^2} = -\frac{5}{16x^2}$$

As before we can try case by case; we end up with $\pm 2\sqrt{x}\,\phi_2' = -\frac{5}{16x^2} \Rightarrow \phi_2' = \mp\frac{5}{32x^{5/2}}$

$$\Rightarrow \phi_2(x) = K \pm \frac{10}{96x^{3/2}}$$

So finally we find that there are 2 solutions so far

$$y(x) = \exp\left(\pm\frac{2}{3}x^{3/2} - \frac{1}{4}\ln x \pm \frac{5}{58x^{3/2}} + K_{\pm}\right)$$

↑ this combines all arbitrary constants so far.

$$= \frac{a}{x^{1/4}}\exp\left(\frac{2}{3}x^{3/2} + o\left(\frac{1}{x^{3/2}}\right)\right)$$

$$+ \frac{b}{x^{1/4}}\exp\left(-\frac{2}{3}x^{3/2} + o\left(\frac{1}{x^{3/2}}\right)\right)$$

Now that we have found a term in the expansion for ϕ_n that $\to 0$ as $x \to \infty$, we can stop expanding, since we know that this contribution vanishes as $x \to \infty$. ! oof !

Example 2 Bessel's equation

$$\frac{d^2 f}{dx^2} + \frac{1}{x}\frac{df}{dx} + \left(1 - \frac{n^2}{x^2}\right) f = 0$$

(Note that here the point at ∞ is regular-singular, so we could also use the Frobenius series to get an expansion of f as $f \to +\infty$).

- First, we need to get rid of the first derivative, so we let

$$y(x) = f(x) e^{\frac{1}{2}\int \frac{1}{x}dx}$$
$$= f(x) e^{\frac{1}{2}\ln x} = f(x)\sqrt{x}$$

Then $\frac{d^2 y}{dx^2} + Q(x) y = 0$ where

$$Q(x) = \left(1 - \frac{n^2}{x^2} + \frac{1}{2x^2} - \frac{1}{4x^2}\right)$$
$$= 1 - \frac{n^2}{x^2} + \frac{1}{4x^2} = 1 - \left(n^2 - \frac{1}{4}\right)\frac{1}{x^2}.$$

- As before assume that $y = \exp(\phi_0 + \phi_1 + \cdots)$ then, to lowest order:

$$\phi_0'' + \phi_0'^2 + 1 - \underbrace{\left(n^2 - \frac{1}{4}\right)\frac{1}{x^2}}_{\text{small compared with 1 so neglect to lowest order}} = 0$$

$\to$ we either have

- $\phi_0'' + 1 = 0 \Rightarrow \phi_0' = -x + C$ not small compared to 1
- $\phi_0'' + \phi_0'^2 = 0 \Rightarrow (\phi_0'(x))^2 = \frac{1}{(x-C)^2}$ — is small compared with 1 (neglected)

$\rightarrow$ finally, $\phi_0'^2 = -1 \Rightarrow \phi_0' = \pm i$ ($\phi_0'' = 0$ so indeed negligible)

$\Rightarrow \phi_0(x) = \pm ix + C$

Starting from there, we now go to the next order:

$$\cancel{\phi_0''} + \phi_1'' + (\phi_0' + \phi_1')^2 + 1 - \left(n^2 - \tfrac{1}{4}\right)\frac{1}{x^2} = 0$$

$$\Rightarrow \cancel{\phi_0''} + \phi_1'' + \cancel{\phi_0'^2} + 2\phi_0'\phi_1' + \cancel{1} - \left(n^2 - \tfrac{1}{4}\right)\frac{1}{x^2} = 0$$

$$\Rightarrow \phi_1'' \pm 2i\,\phi_1' = \left(n^2 - \tfrac{1}{4}\right)\frac{1}{x^2}$$

$$\Rightarrow \text{if } \phi_1' = \varphi \Rightarrow \varphi' \pm 2i\varphi = \left(n^2 - \tfrac{1}{4}\right)\frac{1}{x^2}$$

Let's Try to Create an integrating factor $\mu(x) = e^{\pm 2ix}$

$$\frac{d}{dx}\left(\varphi e^{\pm 2ix}\right) = \frac{e^{\pm 2ix}}{x^2}\cdot\left(n^2 - \tfrac{1}{4}\right)$$

$\rightarrow$ again, this is not easy to integrate

So let's use dominant balance instead:

- If $\varphi' = \left(n^2 - \tfrac{1}{4}\right)\frac{1}{x^2} \Rightarrow \varphi = -\left(n^2 - \tfrac{1}{4}\right)\frac{1}{x}$

  $\rightarrow$ not negligible

- If $\varphi' \pm 2i\varphi = 0 \Rightarrow \varphi = e^{\mp 2ix}$

  $\hookrightarrow$ one is not small the other is exponentially small $\rightarrow$ not useful either

$$\Rightarrow \pm 2i\,\varphi = \left(n^2 - \tfrac{1}{4}\right)\frac{1}{x^2}$$

$$\rightarrow \varphi = \mp \frac{i}{2}\left(n^2 - \tfrac{1}{4}\right)\frac{1}{x^2} = \frac{d\phi_1}{dx}$$

$$\rightarrow \phi_1(x) = \pm \frac{i}{2}\left(n^2 - \tfrac{1}{4}\right)\frac{1}{x} + K \rightarrow 0 \text{ as } x \rightarrow +\infty$$

so we can stop here.

so:

$$y(x) = \exp\left(\pm ix \pm \frac{i}{2}\left(n^2 - \tfrac{1}{4}\right)\frac{1}{x} + K_\pm\right)$$

$$\Rightarrow f(x) = \frac{1}{\sqrt{x}}\left[a\cos\left(x + \mathcal{O}\left(\tfrac{1}{x}\right)\right) + b\sin\left(x + \mathcal{O}\left(\tfrac{1}{x}\right)\right)\right]$$

Note how this appears to be independent of n to this order (see later for more on this).