III A physical interpretation and motivation for the study of boundary layers

Consider the advection diffusion equation (in 1D, here)

$$\frac{\partial f}{\partial t} + v(x)\frac{\partial f}{\partial x} = \varepsilon \frac{\partial^2 f}{\partial x^2} \qquad + \text{ 2 boundary conditions.}$$

$v(x)\frac{\partial f}{\partial x}$: advection term, where $v(x)$ is velocity field

$\varepsilon \frac{\partial^2 f}{\partial x^2}$: diffusion term (assume $\varepsilon > 0$)

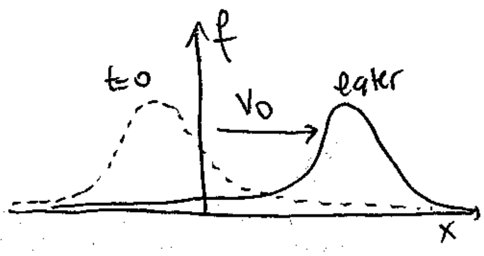
This could represent a number of possible physical systems:

- Advection/diffusion of some chemicals up/down a river / blood vessel / piping by a fluid flow in which case $v(x)$ = velocity of fluid flow (prescribed)
  ε = diffusivity of the chemical
  f = density of the chemical
- Electrophoresis (advection/diffusion of ions/charged particles in an electric field) in which case $v(x) = \mu E(x)$ $E(x)$ = electric field
  f = density of ions
  ε = diffusivity of ions

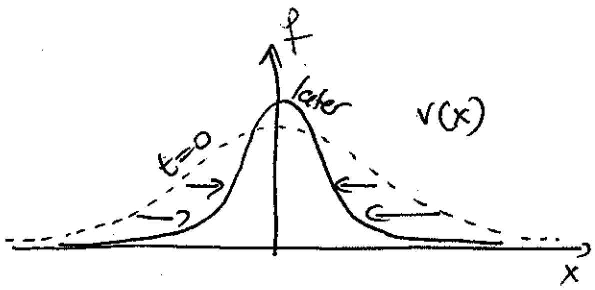
etc...

The behavior of the field $f(x,t)$ depends on the combined effects of the advection & diffusion terms.

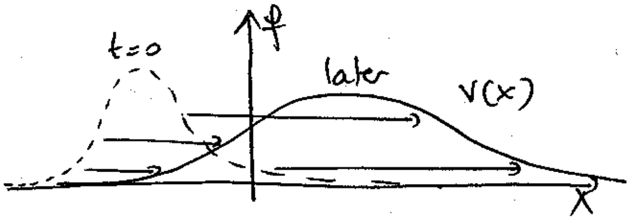
- advection "transports" a field.



advection by a <u>constant</u> velocity merely acts as a translation

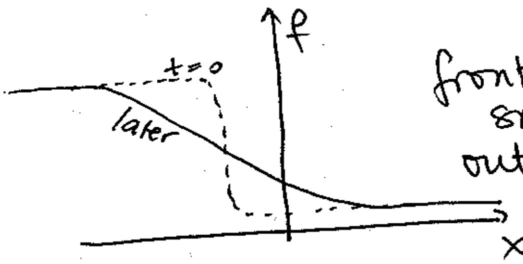


advection by a convergent velocity field "compresses" the function $f(x,t)$

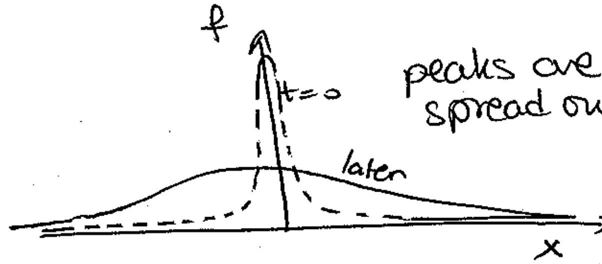


advection by a divergent velocity field can "expand" the function $f(x,t)$

- diffusion. acts to smooth out strong gradients



fronts are smoothed out



peaks are spread out

In a steady state, $\frac{\partial f}{\partial t} = 0$ & the function $f(x, t \to \infty)$ results from a balance between advection & diffusion. It is the solution of $v(x)\frac{df}{dx} = \varepsilon\frac{d^2f}{dx^2}$

or in other words, $\underbrace{\varepsilon\frac{d^2f}{dx^2} - v(x)\frac{df}{dx} = 0}_{\text{an ODE that has a singular behaviour as } \varepsilon \to 0\,!}$

Conversedly, any ODE of the form $\varepsilon\frac{d^2f}{dx^2} + g(x)\frac{df}{dx} = 0$ can be interpreted as the steady-state result of an advection/diffusion process $\to$ this will be very useful when trying to determine, in general, where the boundary layer is (see below).

Before proceeding, note that it's very easy to generalize the analogy with any 2nd order ODE of the form

$$\varepsilon \frac{d^2 f}{dx^2} + g(x)\frac{df}{dx} + h(x) f = 0$$

→ Simply rewrite this as a PDE & interpret the terms:

$$\frac{\partial f}{\partial t} - g(x)\frac{df}{dx} - h(x) f = \varepsilon \frac{\partial^2 f}{\partial x^2}$$

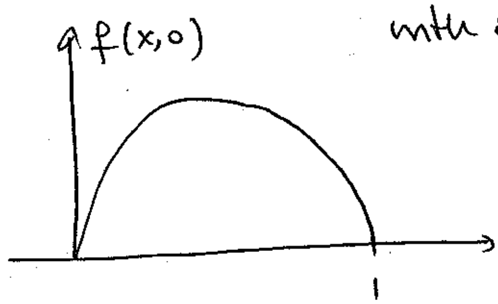
$g(x)$ is − the advection velocity

$h(x)$ is clearly an amplification/damping term: if $h(x) > 0$ f grows exponentially
if $h(x) < 0$ f is damped.

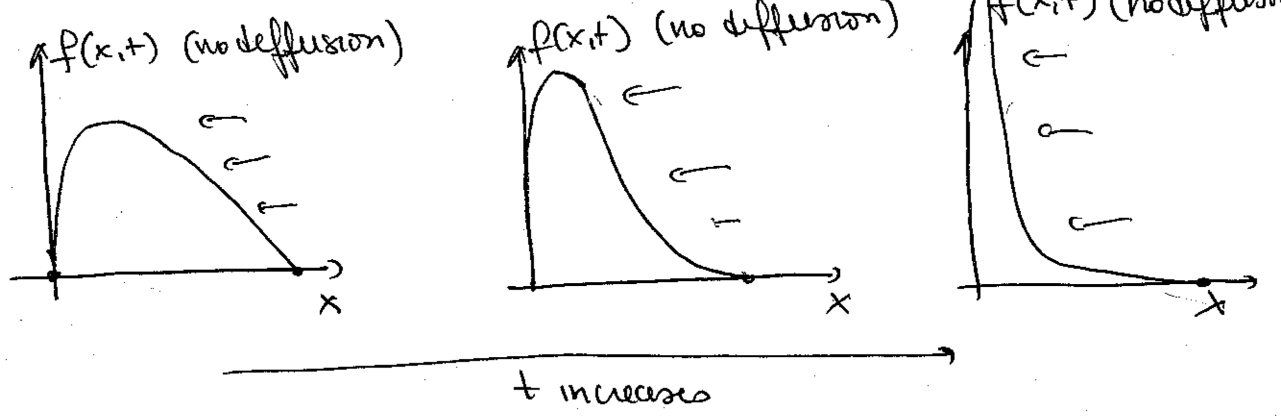
Let's now look at a few simple examples and see if we can "guess" where the boundary layer will be.

<u>Example 1</u> $\quad \dfrac{\partial f}{\partial t} - \dfrac{\partial f}{\partial x} = \varepsilon \dfrac{\partial^2 f}{\partial x^2}$ with $\varepsilon > 0$

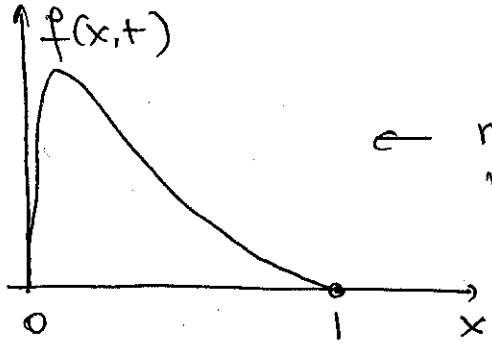
$f(0) = 0$
$f(1) = 0$
$f(x,0) = x(1-x)$



First inspect what the advection term would do:
→ wants to transport the entire parabola to the left. However, the boundary condition at $x=0$ constrains $f(0)=0$ → without diffusion the whole $f(x,t)$ field would just pile up near $x=0$



Diffusion, however, would prevent the generation of sharp gradients so we expect, in steady state balance, something like:



← now we clearly see where the boundary layer is: near $x=0$.

<u>Example 2</u> Consider the ODE $\varepsilon \frac{d^2f}{dx^2} + 2\left(\frac{df}{dx}\right)\left(x-\frac{1}{2}\right) = 0$

with $\varepsilon > 0$, and $f(0) = 0$ $f(1) = 0$

Where is the boundary layer?

→ to answer this question, let's recast this as an advection-diffusion problem:

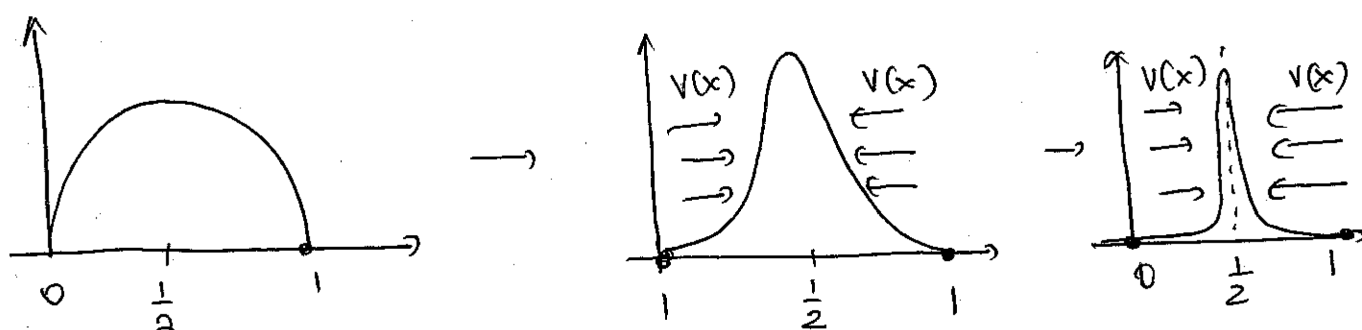
$$\frac{\partial f}{\partial t} - 2\left(x-\frac{1}{2}\right)\frac{\partial f}{\partial x} = \varepsilon \frac{\partial^2 f}{\partial x^2}$$

→ this has a velocity field $v(x) = 2\left(x-\frac{1}{2}\right)$

so $v(x) > 0$ if $x < \frac{1}{2}$

$v(x) < 0$ if $x > \frac{1}{2}$

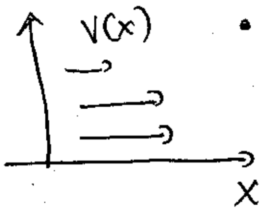
so, supposing we start with the same initial conditions as before, we'd get



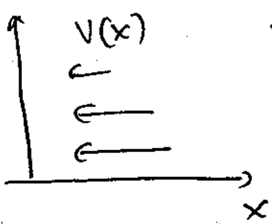
→ The advection field tends to let $f(x,t)$ <u>pile up</u> near $x = \frac{1}{2}$. While diffusion would try to prevent this we see that this is the cause of a boundary layer. In this case, the "boundary" layer is actually detached from the physical boundaries of the system ⇒ this is called an "internal" boundary layer.

These arguments can easily be formalized, if necessary, by solving for the non-diffusive, time-dependent problem using the method of characteristics. We will see an example later. In the meantime let's summarize our findings so far in the more general case:

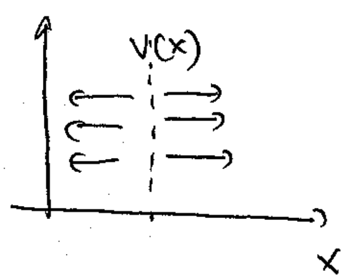
Given the equation $\varepsilon \frac{d^2 f}{dx^2} - V(x) \frac{df}{dx} + h(x) f = 0$, with $\varepsilon > 0$



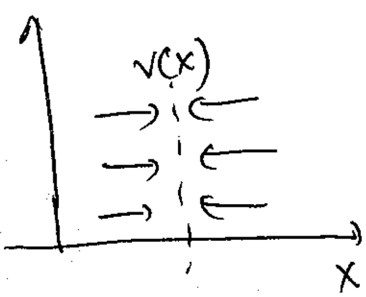
- if $V(x) > 0$ in the interval considered then the boundary layer (if it exists) must be on the right.



- If $V(x) < 0$ in the interval considered, then the boundary layer (if it exists) must be on the left.
- If $v(x)$ changes sign once within the interval considered, then the result depends on the derivative of $V(x)$ too:



— if $V(x)$ goes from negative to positive, then there may in fact be two boundary layers (one on the left & one on the right)



— if $V(x)$ goes from positive to negative then there will be an internal boundary layer at the point where $V(x) = 0$.

Notes: • In all we had so far, we required $\varepsilon > 0$. A.
If $\varepsilon < 0$, then create another small parameter (say η) where $\eta = -\varepsilon$ so that $\eta > 0$ and proceed from there.

• If the ODE is nonlinear in the first or second derivative terms, then none of this really applies as is & one must be much more careful about the determination of the location of the BL.
$\Rightarrow$ Either do it (if possible) using the method of characteristics (which tell you of the direction of propagation of information) or do it numerically

IV Boundary layer thickness & the principle of least degeneracy

Now that we know where to find the boundary layer, we wish to know how its thickness may scale with ε.

We have already learned of a possible method - the method of dominant balance. We now learn of an alternative, called the principle of least degeneracy.

The idea is the following:

- Find the 0^{th} order outer expansion
- Suppose that the correct scaled/stretched variable in the inner expansion is $s = \frac{X - X_{bl}}{\varepsilon^p}$ where X_{bl} is the position of the bl, and p is to be decided

- Solve for the 0^{th} order inner expansion & see if there is any hope of matching it to the outer one.

As we will see, the last step usually requires the inner expansion to have enough "complexity" to match the outer; in other words, the equation obtained for the inner expansion must be complex enough to have a solution that could match the outer $\rightarrow$ this is the Van Dyke principle of least degeneracy.

Example 1 Consider the same equation we had earlier:

$$\varepsilon \frac{d^2 f}{dx^2} + \frac{df}{dx} + f = 0 \qquad \begin{array}{l} f(0)=0 \\ f(1)=1 \end{array}$$

We found that the outer expansion was:

$$f_{outer} = e^{1-x} + \varepsilon\left(e^{1-x} - x e^{1-x}\right) + h.o.t$$

$$= e^{1-x} + \varepsilon(1-x)e^{1-x}$$

$$= e^{1-x} \quad (1+\varepsilon(1-x)) + h.o.t$$

let's now suppose that the correct stretched variable for the inner expansion is $s = \frac{x}{\varepsilon^p}$ where p is to be determined.

then $\frac{d}{dx} = \frac{1}{\varepsilon^p}\frac{d}{ds} \Rightarrow \frac{d^2}{dx^2} = \frac{1}{\varepsilon^{2p}}\frac{d^2}{ds^2}$

so the equation becomes

$$\varepsilon^{1-2p}\frac{d^2 f}{ds^2} + \varepsilon^{-p}\frac{df}{ds} + f = 0$$

Unless $p<0$, we see that the term in ε^{-p} is always larger than the one in ε^0. Since $p<0$

would be inconsistent with the assumption that we have a boundary layer, we can rule it out. This implies that in all generality, we always have, in the b.l., and to lowest-order in the expansion,

$$\varepsilon^{1-2p} \frac{d^2 f_0}{ds^2} + \varepsilon^{-p} \frac{df_0}{ds} \simeq 0$$

This then leaves 3 options

- either $\varepsilon^{1-2p} = o(\varepsilon^{-p})$ in which case the inner expansion boils down to

$$\frac{df_0}{ds} = 0$$

- or $\varepsilon^{-p} = o(\varepsilon^{1-2p})$ in which case the inner expansion boils down to

$$\frac{d^2 f_0}{ds^2} = 0$$

- or $\varepsilon^{1-2p} = \varepsilon^{-p}$ in which case

$$\frac{d^2 f_0}{ds^2} + \frac{df_0}{ds} = 0 .$$

Suppose we pick the first case:

$$\frac{df_0}{ds} = 0, \quad \text{with } f_0(0) = 0 \Rightarrow f_0(s) = 0 \quad \forall s$$

→ there is no way in which this can be matched to the outer expansion since $\lim_{x \to 0} f_{outer} = 1 + \varepsilon\ \text{hot}$

Suppose we pick the second case:

$$\frac{d^2 f_0}{ds^2} = 0 \quad \text{with } f_0(0) = 0 \Rightarrow f_0(s) = As$$

→ Again we cannot match this to the outer because there is no way in which we can pick A such that $\lim_{s \to \infty} f_0(s)$ is a constant

$\Rightarrow$ only the last possibility works (see previous section).

This illustrates that if the inner region governing equation becomes too simple, its solution will also be too simple to exhibit the required behaviour for matching onto the outer

least Degeneracy means "we don't want the inner expansion to become too "degenerate"."

Another way of looking at this problem is to note that we need to keep all 3 terms in the matching region :

inner region $\longleftrightarrow$ $\longleftarrow$ outer region $\longrightarrow$ $\leftarrow$ From dominant balance

$$\underbrace{\varepsilon \frac{d^2 f}{dx^2} \sim \frac{df}{dx}}_{\text{inner region}} \qquad \underbrace{\frac{df}{dx} \sim f}_{\text{outer region}}$$

So in the overlap region we <u>need</u> all 3 terms to be important : $\varepsilon \frac{d^2 f}{dx^2} \sim \frac{df}{dx} \sim f$

If , on the other hand, the inner region only has one dominant term, then we cannot do the matching.

Inner region $\longleftrightarrow$ outer region $\longleftrightarrow$

$$\varepsilon \frac{d^2 f}{dx^2} \qquad ? \qquad \frac{df}{dx} \sim f$$

In this example, we already knew the b.l. thickness. Let's now study another, less obvious example.