In the previous Chapter, we began to study problems that exhibit boundary layers, which are regions close to the boundary where a function varies much more rapidly than in the bulk of the domain, usually as a result of a near-singularity in the governing ODE (i.e., ϵ multiplying the highest derivative).

In this Chapter, we will study an alternative method of obtaining the solution that does not require matched asymptotic expansions". This method is called a multiscale expansion. First, however, we must learn a simple method for identifying the behavior of the solution in & out of the boundary layer.

I The method of dominant balance

Consider again the two system studied in the previous Chapter:

$$① \begin{cases} \epsilon \dfrac{dy}{dx} + y = e^{-x} \\ y(0) = 2 \end{cases} \qquad ② \begin{cases} \epsilon \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = 0 \\ y(0) = 0 \quad y(1) = 1 \end{cases}$$

The key to the method of dominant balance is to
① ignore the BCs
② Ask the question: suppose this equation can be simplified so that only 2 terms are left that balance each other → what solution

does this imply & is it consistent with the simplification made?

Example 1

① $\varepsilon \frac{dy}{dx} + y = e^{-x}$

→ 3 possibilities

$\varepsilon \frac{dy}{dx} + y \sim 0 \quad \longrightarrow \quad y \sim Ke^{-\frac{x}{\varepsilon}}$

$\varepsilon \frac{dy}{dx} \sim e^{-x} \quad \longrightarrow \quad y \sim -\frac{1}{\varepsilon} e^{-x} + K$

$y \sim e^{-x} \quad \longrightarrow \quad y \sim e^{-x}$

Now check if the neglected term is indeed negligible:

Case 1: compare $Ke^{-\frac{x}{\varepsilon}}$ with e^{-x}

→ this is indeed such that $Ke^{-\frac{x}{\varepsilon}} \gg e^{-x}$ only when x is $\ll \varepsilon$. Otherwise, the solution cannot exist

Case 2: $-\frac{1}{\varepsilon} e^{-x} + K$ is clearly always $\gg e^{-x}$

⇒ so we were not justified in neglecting y.

→ this balance cannot exist

Case 3: if $y \sim e^{-x}$ then $\varepsilon \frac{dy}{dx} \sim -\varepsilon e^{-x}$ which is indeed $\ll e^{-x}$.

→ this solution is always a possibility

Summary: this system can either have a dominant balance with $y \sim e^{-x}$ (always) or one with $\varepsilon \frac{dy}{dx} + y \sim 0$ (but only for small x)

→ This tells you not only where the Boundary Layer must be (small x) but also what its shape is!

Example 2

② $\varepsilon \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$

Case 1 : $\varepsilon \frac{d^2y}{dx^2} + \frac{dy}{dx} \sim 0$

$\rightarrow$ $y \sim Ae^{-\frac{x}{\varepsilon}} + K$

$\rightarrow$ $\frac{dy}{dx} \sim -\frac{A}{\varepsilon} e^{-\frac{x}{\varepsilon}}$

$\frac{d^2y}{dx^2} \sim +\frac{A}{\varepsilon^2} e^{-\frac{x}{\varepsilon}}$

So clearly $\varepsilon \frac{d^2y}{dx^2} \sim \frac{dy}{dx} \gg y$. for small x. $(e^{-\frac{x}{\varepsilon}} \gg 1)$

For large x, we need to worry about the value of K.

Case 2 : $\varepsilon \frac{d^2y}{dx^2} + y \equiv 0$

$\rightarrow$ $y \sim A\cos\left(\frac{x}{\sqrt{\varepsilon}}\right) + B\sin\left(\frac{x}{\sqrt{\varepsilon}}\right)$

$\rightarrow$ $\frac{dy}{dx} \sim -\frac{A}{\sqrt{\varepsilon}} \sin\left(\frac{x}{\sqrt{\varepsilon}}\right) + \frac{B}{\sqrt{\varepsilon}} \cos\left(\frac{x}{\sqrt{\varepsilon}}\right)$

$\rightarrow$ here we immediately see that $\frac{dy}{dx} \gg y$,

So we should not have neglected it

$\rightarrow$ thus balance cannot exist

Case 3 : $\frac{dy}{dx} + y = 0$

$\rightarrow$ $y = Ae^{-x}$ $\rightarrow$ $\frac{dy}{dx} = -Ae^{-x}$

$\rightarrow$ $\varepsilon \frac{d^2y}{dx^2} = \varepsilon Ae^{-x}$

$\Rightarrow$ indeed, $\varepsilon \frac{d^2y}{dx^2}$ is negligible

$\rightarrow$ thus balance is always possible

Summary : we can either have $\frac{dy}{dx} + y \simeq 0$ (always)

or $\varepsilon \frac{d^2y}{dx^2} + \frac{dy}{dx} \sim 0$ (for small x)

$\rightarrow$ again this tells us where the bl is, and that

$y \sim e^{-x/\varepsilon}$ in the bl!

We will see more of the method of dominant balance in upcoming Chapters. Note that its use is not limited to ODES; we can also use it for polynomial/algebraic equations and for PDES.

II Inner & outer expansions in a system with boundary layers

A The outer expansion

The outer expansion, by definition, is an expansion that is uniformly valid outside of the boundary layer. We obtain it simply by doing a standard expansion & fit whatever boundary conditions are not located in the boundary layer.

Example 1 $\varepsilon \dfrac{dy}{dx} + y = e^{-x}$, $y(0)=2$

let $y = y_0 + \varepsilon y_1 + \cdots$

$\Rightarrow$ $y_0 = e^{-x}$ at $O(\varepsilon^0)$

$\dfrac{dy_0}{dx} + y_1 = 0$ at $O(\varepsilon^1)$

$\Rightarrow y_1 = -\dfrac{dy_0}{dx} = -\left(-e^{-x}\right) = e^{-x}$

etc.

Here, there are no boundary conditions outside the boundary layer, which is convenient since there are no arbitrary integration constan appearing in the expansion.

Example 2 : $\varepsilon \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$ $\quad y(0)=0$, $y(1)=1$

$\Rightarrow y = y_0 + \varepsilon y_1 + \cdots$

$\Rightarrow O(\varepsilon^0): \frac{dy_0}{dx} + y_0 = 0 \rightarrow y_0(x) = Ke^{-x}$

$y_0(1) = 1 \Rightarrow K = e$

so $y_0(x) = e^{1-x}$

$O(\varepsilon)$: $\frac{d^2y_0}{dx^2} + \frac{dy_1}{dx} + y_1 = 0$

$\Rightarrow \frac{dy_1}{dx} + y_1 = -\frac{d^2y_0}{dx^2} = -Ke^{-x} = -e^{1-x}$

$\Rightarrow y_1(x) = Ae^{-x} + Bxe^{-x}$

where B satisfies:

$Be^{-x} - \cancel{Bxe^{-x}} + \cancel{Bxe^{-x}} = -e^{1-x}$

$\Rightarrow Be^{-x} = -e.e^{-x}$

$\Rightarrow B = -e$

and A is given by $y_1(1) = 0 \Rightarrow$

$y_1(1) = 0 = Ae^{-1} - ee^{-1}$

$\Rightarrow A = e$ so

$y_1(x) = e^{1-x} - xe^{1-x} = (1-x)e^{1-x}.$

⋮

etc

Note that here we are less worried about non-uniformity : the interval $[0,1]$ is bounded and these functions are non-singular in $[0,1]$ $\Rightarrow$ the expansion is uniform by construction.

(B) The inner expansion → this is the expansion in the bl

The method of dominant balance suggested that, in the boundary layer and for both examples,

$$y \sim e^{-x/\varepsilon}$$

→ this suggests that the system varies on the fast variable $\frac{x}{\varepsilon}$ rather than x.

→ Introduce the stretched coordinate $s = \frac{x}{\varepsilon}$.

then $\frac{d}{dx} = \frac{ds}{dx}\frac{d}{ds} = \frac{1}{\varepsilon}\frac{d}{ds}$. Note that s is $O(1)$ when x is $O(\varepsilon)$.

⇒ Use this instead of x to get the correct expansion, solve the equation & match to inner BC

Example : (accidental switch. Read Example 1 below first)

$$\varepsilon\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0 \rightarrow \frac{\varepsilon}{\varepsilon^2}\frac{d^2y}{ds^2} + \frac{1}{\varepsilon}\frac{dy}{ds} + y = 0$$

$$\rightarrow \frac{1}{\varepsilon}\left(\frac{d^2y}{ds^2} + \frac{dy}{ds}\right) + y = 0$$

Now let $y = y_0 + \varepsilon y_1 + \cdots$ as usual.

At $O(\varepsilon^{-1})$ we get

$$\frac{d^2y_0}{ds^2} + \frac{dy_0}{ds} = 0 \rightarrow y_0(s) = A_0 + B_0e^{-s}$$

At $O(\varepsilon^0)$ we get

$$\frac{d^2y_1}{ds^2} + \frac{dy_1}{ds} + y_0 = 0$$

$$\Rightarrow \frac{d^2y_1}{ds^2} + \frac{dy_1}{ds} + A_0 + B_0e^{-s} = 0$$

$$\Rightarrow \frac{d^2y_1}{ds^2} + \frac{dy_1}{ds} = -A_0 - B_0e^{-s}.$$

The solution to this is

$$y_1(s) = A_1 e^{-s} + B_1 - A_0 s + K s e^{-s}$$

where K satisfies

$$-2Ke^{-s} + \cancel{Kse^{-s}} + Ke^{-s} - \cancel{Kse^{-s}} = -B_0 e^{-s}$$

$$\Rightarrow \quad K = B_0$$

so $y_1(s) = A_1 e^{-s} + B_1 - A_0 s + B_0 s e^{-s}$

To fit the inner boundary conditions, $y(0)=0$, requires $y_0(0)=0$

$$y_1(0) = 0$$

The first implies $A_0 + B_0 = 0 \Rightarrow A_0 = -B_0$

so $y_0(s) = A_0(1-e^{-s})$

The second implies

$$A_1 + B_1 = 0 \Rightarrow B_1 = -A_1$$

so $y_1(s) = A_1(e^{-s}-1) - A_0 s(1+e^{-s})$

The remaining constants, A_0 and A_1 come from matching the inner expansion to the outer one.

Summary so far for example 2:

<u>Inner exp:</u> $y_I(s) = A_0(1-e^{-s}) + \varepsilon A_1(e^{-s}-1) - \varepsilon A_0 s(1+e^{-s}) + \dots$

<u>Outer exp:</u> $y_0(x) = e^{1-x} + \varepsilon(1-x)e^{1-x} + \dots$

<u>Example 1</u> $\quad \varepsilon \frac{dy}{dx} + y = e^{-x} \quad y(0)=2$

$$\Rightarrow \quad \frac{\varepsilon}{\varepsilon}\frac{dy}{ds} + y = e^{-\varepsilon s}$$

$$\Rightarrow \quad \frac{dy}{ds} + y = 1 - \varepsilon s + \frac{\varepsilon^2 s^2}{2} - \dots$$

since s is $O(1)$ in the bl.

letting $y = y_0 + \varepsilon y_1 + \cdots$

→ to lowest order

$$\begin{cases} \frac{dy_0}{ds} + y_0 = 1 \\ y_0(0) = 2 \end{cases} \rightarrow \quad y_0(s) = A_0 e^{-s} + 1$$

To impose the BC,

$2 = A_0 + 1 \Rightarrow A_0 = 1$

so $y_0(s) = e^{-s} + 1$

→ to $O(\varepsilon)$:

$$\begin{cases} \frac{dy_1}{ds} + y_1 = -s \\ y_1(0) = 0 \end{cases} \rightarrow \quad y_1(s) = A_1 e^{-s} + a_1 s + b_1$$

a_1 & b_1 satisfy

$a_1 + a_1 s + b_1 = -s$

therefore $a_1 + b_1 = 0$

$a_1 = -1 \Rightarrow b_1 = 1$

So $y_1(s) = A_1 e^{-s} + 1 - s$

To satisfy the BC: $0 = A_1 + 1 \Rightarrow A_1 = -1$

So $y_1(s) = 1 - e^{-s} - s$

$$\Rightarrow \quad y(s) = e^{-s} + 1 + \varepsilon(1 - e^{-s} - s) + \cdots$$

In this case there no longer appears to be any leftover constant ⇒ we therefore hope that the inner & outer expansions match "as is".

(c) Prandtl's matching condition

Prandtl proposed that for the expansion to be consistent, we need to have that the limit of the inner expansion as $s \to \infty$ is equal to that of the outer expansion as $x \to 0$ (if the bl is near $x=0$).

or in other words, the boundary layer solution must smoothly match onto the outer expansion as one moves out of the boundary layer, and the outer expansion must smoothly match onto the inner expansion moving into the boundary layer

$$\Rightarrow \quad \lim_{x \to 0} y_{outer}(x) = \lim_{s \to \infty} y_{inner}(s)$$

Example 1. In this example, we merely have to verify that this is true:

We had: $y_{outer}(x) = e^{-x} + \varepsilon e^{-x} + \cdots$

$$y_{inner}(s) = e^{-s} + 1 + \varepsilon(1 - e^{-s} - s) + \cdots$$

$$\lim_{x \to 0} y_{outer} = \lim_{x \to 0} (1 + \varepsilon + \cdots) e^{-x}$$

$$= (1 + \varepsilon + \cdots)(1 - x + \frac{x^2}{2} + \cdots)$$

$$= 1 + \varepsilon - x - \varepsilon x \cdots$$

$$\lim_{s \to \infty} y_{inner} = 1 + \varepsilon - \varepsilon s + \cdots$$

Are these the same? Recall that $s = \frac{x}{\varepsilon}$ so

$$\lim_{s \to \infty} y_{inner} = 1 + \varepsilon - x + \cdots$$

$\to$ to lowest order, the two match and are equal to $1 - x \simeq 1$ as $x \to 0$

Note that Prandtl's matching condition only works for 1-term expansions (lowest order). See later for higher-order matching

Example 2

$$\lim_{x\to 0} y_{outer}(x) = \lim_{x\to 0} e^{1-x} + \varepsilon(1-x)e^{1-x} + \cdots$$

$$= \lim_{x\to 0} e\left(1 - x + \frac{x^2}{2} + \cdots\right) + \varepsilon(1-x)e\left(1-x+\frac{x^2}{2}+\cdots\right)$$

$$\lim_{s\to\infty} = y_{inner}(s)$$

$$= \lim_{s\to\infty} A_0(1-e^{-s}) + \varepsilon A_1(e^{-s}-1) - \varepsilon A_0 s(e^{-s}+1)$$

$$= \lim_{s\to\infty} A_0 - \varepsilon A_1 - \varepsilon A_0 s + \cdots$$

Matching the two to lowest order in ε and in the limit $x \to 0$ yields $A_0 = e$ straightforwardly. At the next order we need to be more careful, and different methods must be used.

→ But to lowest order, we therefore have

$$y_{outer}(x) = e^{1-x}$$

$$y_{inner}(x) = e(1-e^{-s}) = e(1-e^{-\frac{x}{\varepsilon}})$$

(D) The composite expansion

A single expression combining both limits can then be obtained as follows

$$y_{composite}(x) = y_{inner}(x) + y_{outer}(x) - \lim_{x\to 0} y_{outer}(x)$$

or alternatively

$$y_{composite}(x) = y_{inner}(x) + y_{outer}(x) - \lim_{s\to\infty} y_{inner}(s)$$

Let's see what this gives us:

Example 1

$$y_{composite}(x) = y_{outer}(x) + y_{inner}(x) - \lim_{x \to 0} y_{outer}(x)$$
$$= e^{-x} + e^{-x/\varepsilon} + 1 - 1$$
$$= e^{-x} + e^{-x/\varepsilon} \quad \checkmark \text{ as before!}$$

Example 2

$$y_{composite}(x) = y_{outer}(x) + y_{inner}(x) - \lim_{x \to 0} y_{outer}(x)$$
$$= e^{1-x} + e\left(1 - e^{-x/\varepsilon}\right) - e$$
$$= e^{1-x} + \cancel{e} - e^{1-x/\varepsilon} - \cancel{e}$$
$$= e^{1-x} - e^{1-x/\varepsilon}$$

→ more or less as before
(we had $e^{1-x} - e^{1+x-\frac{x}{\varepsilon}}$ but $x \ll \frac{x}{\varepsilon}$ so its consisten

(E) Summary of the method

1. Use dominant balance to identify possible behaviors of solution & if possible, where & what scale is the boundary layer
2. Solve for the outer expansion, match BCs at that boundary.
3. Stretch coordinate, solve for inner expansion, match BCs at this boundary
4. Find remaining integrating constants by matching inner & outer expansions
5. Create composite expansion.