Problem 1

(i) $\varepsilon \frac{d^2f}{dx^2} + (2+x)\frac{df}{dx} + f = 1 \qquad \varepsilon > 0$

$f(0) = 2 \qquad f(1) = 0$

- velocity field is $< 0 \Rightarrow$ bl is on the left. $(x=0)$
- outer solution: let $f = f_0 + \varepsilon f_1 + \cdots$

⊕ to 0th order: $(2+x)\frac{df_0}{dx} + f_0 = 1$

$$\Rightarrow \frac{d}{dx}\left((2+x)f_0\right) = 1$$

$$\Rightarrow f_0(x) = \frac{x+K}{2+x}$$

we apply the BC @ $x=1$: $f_0(1) = 0 \Rightarrow$

$$\frac{1+K}{3} = 0 \Rightarrow K = -1$$

so $f_0(x) = \frac{x-1}{2+x}$.

⊕ to 1st order:

$$\frac{d^2f_0}{dx^2} + (2+x)\frac{df_1}{dx} + f_1 = 0$$

$$\Rightarrow \frac{d}{dx}\left((2+x)f_1\right) = -\frac{d^2}{dx^2}\left(\frac{x-1}{2+x}\right)$$

$$\Rightarrow (2+x)f_1 = -\frac{d}{dx}\left(\frac{x-1}{2+x}\right) + C$$

$$= -\left[\frac{1}{2+x} - \frac{x-1}{(2+x)^2}\right] + C$$

$f_1(1) = 0 \Rightarrow 0 = -\left[\frac{1}{3}\right] + C \Rightarrow C = \frac{1}{3}$

So $f_1(x) = \frac{1}{3(2+x)} - \frac{1}{(2+x)^2} + \frac{x-1}{(2+x)^3}$

$$= \frac{1}{3(2+x)} - \frac{1}{(2+x)^2}\left(1 - \frac{x-1}{2+x}\right)$$

$$\Rightarrow \quad f_1(x) = \frac{1}{3(2+x)} - \frac{3}{(2+x)^3}$$

- <u>Inner solution:</u>

let $\quad s = \frac{x}{\varepsilon^p} \quad$ then $\quad x = \varepsilon^p s$

$$\varepsilon^{1-2p} \frac{d^2 f}{ds^2} + (2 + \varepsilon^p s)\varepsilon^{-p} \frac{df}{ds} + f = 1$$

$$\Rightarrow \quad \text{need} \quad 1-2p = -p \Rightarrow p = 1$$

let $\quad f = f_0 + \varepsilon f_1 + \cdots$

- so to order $\frac{1}{\varepsilon}$:

$$\frac{d^2 f_0}{ds^2} + 2\frac{df_0}{ds} = 0 \quad \Rightarrow \quad f_0(s) = A_0 e^{-2s} + B_0$$

Applying BC @ $x=0$: $f_0(0) = 2$: $\quad A_0 + B_0 = 2$
$\rightarrow \quad A_0 = 2 - B_0$

$$f_0(s) = (2-B_0)e^{-2s} + B_0$$

- to order 1:

$$\frac{d^2 f_1}{ds^2} + 2\frac{df_1}{ds} + s\frac{df_0}{ds} + f_0 = 1$$

$$\Rightarrow \quad \frac{d^2 f_1}{ds^2} + 2\frac{df_1}{ds} = 1 - f_0 - s\frac{df_0}{ds}$$

$$= 1 - B_0 - (2-B_0)e^{-2s} + 2s(2-B_0)e^{-2s}$$

$$= 1 - B_0 - (2-B_0)(1-2s)e^{-2s}$$

$\rightarrow$ solution is of the form

$$f_1(s) = A_1 e^{-2s} + B_1 + \underbrace{C_1 s + C_2 s e^{-2s} + C_3 s^2 e^{-2s}}_{\text{particular solution.}}$$

$$\rightarrow \quad \frac{df_1^{ps}}{ds} = C_1 + C_2 e^{-2s} - 2C_2 s e^{-2s} + 2C_3 s e^{-2s} - 2C_3 s^2 e^{-2s}$$

$$\frac{d^2 f_1^{ps}}{ds^2} = -4C_2 e^{-2s} + 4C_2 s e^{-2s} + 2C_3 e^{-2s} - 8C_3 s e^{-2s} + 4C_3 s^2 e^{-2s}$$

So

$$\cancel{4C_3s^2e^{-2s}} - 8C_3se^{-2s} + 2C_3e^{-2s} + \cancel{4C_2se^{-2s}} - 4C_2e^{-2s}$$
$$+ 2C_1 + 2C_2e^{-2s} - \cancel{4C_2se^{-2s}} + 4C_3se^{-2s} - \cancel{4C_3s^2e^{-2s}}$$
$$= 1-B_0 - (2-B_0)e^{-2s} + 2s(2-B_0)e^{-2s}$$

$$\Rightarrow \begin{cases} 2C_1 = 1-B_0 & \Rightarrow C_1 = \frac{1-B_0}{2} \\ 2C_3 - 4C_2 + 2C_2 = -(2-B_0) & \Rightarrow C_2 = 0 \\ -8C_3 + 4C_3 = 2(2-B_0) & \Rightarrow C_3 = -\frac{1}{2}(2-B_0) \end{cases}$$

So $\quad f_1(s) = A_1e^{-2s} + B_1 + \frac{1-B_0}{2}s - \frac{1}{2}(2-B_0)s^2e^{-2s}$

$f_1(0) = 0 \Rightarrow A_1 + B_1 = 0 \Rightarrow A_1 = -B_1$

- matching:

let's first Taylor-expand the outer near $x=0$:

$$f^{outer}(x) = \frac{x-1}{2+x} + \varepsilon\left[\frac{1}{3(2+x)} - \frac{3}{(2+x)^3}\right] + \cdots$$
$$\simeq \frac{x-1}{2}\left(1 - \frac{x}{2} + \frac{x^2}{4}\cdots\right) + \varepsilon\left[\frac{1}{6}\left(1 - \frac{x}{2} + \frac{x^2}{4}\cdots\right) - \frac{3}{8}\left(1 - \frac{3x}{2} + \cdots\right)\right]$$
$$\simeq -\frac{1}{2} + \frac{3x}{4} + \cdots + \varepsilon\left(\frac{1}{6} - \frac{3}{8} + \cdots\right)$$

Now let $x = \varepsilon s$:

$$= -\frac{1}{2} + \frac{3}{4}\varepsilon s - \frac{10}{48}\varepsilon + o(\varepsilon^2)$$

The inner as $s \to +\infty$ is:

$$\lim_{s\to+\infty} f^{inner}(s) = B_0 + \varepsilon\left(B_1 + \frac{1-B_0}{2}s\right)$$

Matching $\Rightarrow$ $B_0 = -\frac{1}{2}$

$$B_1 + \cancel{\frac{1-B_0}{2}s} = -\frac{5}{24} + \cancel{\frac{3}{4}s}$$
$$\Rightarrow B_1 = -\frac{5}{24}$$

$$\Rightarrow f^{inner}(s) = \frac{5}{2}e^{-2s} - \frac{1}{2} + \varepsilon\left[\frac{5}{24}(e^{-2s}-1) + \frac{3}{4}s - \frac{5}{4}s^2e^{-2s}\right]$$

Finally, the composite expansion is

$$f_{compo}(x) = f^{outer}(x) + f^{inner}(x) - \underset{\text{common limit}}{L}$$

$$= \frac{x-1}{2+x} + \varepsilon\left[\frac{1}{3(2+x)} - \frac{3}{(2+x)^3}\right]$$

$$+ \frac{5}{2}e^{-2\frac{x}{\varepsilon}} - \cancel{\tfrac{1}{2}} + \varepsilon\left[\frac{5}{24}\left(e^{-2\frac{x}{\varepsilon}} \cancel{-1}\right) + \cancel{\frac{3}{4}\frac{x}{\varepsilon}} - \frac{5}{4}\frac{x^2}{\varepsilon^2}e^{-\frac{2x}{\varepsilon}}\right]$$

$$-\left(\cancel{-\tfrac{1}{2}} + \varepsilon\left(\cancel{-\tfrac{5}{24}} + \cancel{\frac{3}{4}\frac{x}{\varepsilon}}\right)\right)$$

$$= \frac{x-1}{2+x} + \varepsilon\left[\frac{1}{3(2+x)} - \frac{3}{(2+x)^3}\right]$$

$$+ \frac{5}{2}e^{-2\frac{x}{\varepsilon}} + \varepsilon\left[\frac{5}{24}e^{-2\frac{x}{\varepsilon}} - \frac{5}{4}\frac{x^2}{\varepsilon^2}e^{-\frac{2x}{\varepsilon}}\right]$$ as required

(iii). $\varepsilon\frac{d^2f}{dx^2} + (x-2)\frac{df}{dx} + f = 2-x \qquad \varepsilon > 0$

- velocity field is positive on $(0,1) \Rightarrow$ bl is on right. $(x=1)$
- outer solution: let $f = f_0 + \varepsilon f_1 + \dots$

<u>0th ord</u>: $(x-2)\frac{df_0}{dx} + f_0 = 2-x$

$$\Rightarrow \quad \frac{d}{dx}\left(f_0(x)(x-2)\right) = 2-x$$

$$\Rightarrow \quad f_0 = \frac{2x - \frac{x^2}{2} + c}{x-2}$$

BC @ $x=0 \Rightarrow 2 = \frac{c}{-2} \Rightarrow c = -4$

So $f_0(x) = \dfrac{-4+2x-\frac{x^2}{2}}{x-2} = 2 - \dfrac{x^2}{2x-4}$

$$\left(= \frac{2-x}{2} + \frac{2}{2-x}\right)$$

<u>1st ord</u>: $\frac{d^2f_0}{dx^2} + (x-2)\frac{df_1}{dx} + f_1 = 0$

$$\Rightarrow \quad \frac{d}{dx}\left((x-2)f_1\right) = -\frac{d^2f_0}{dx^2}$$

$$(x-2)f_1 = -\frac{df_0}{dx} + c$$

$$\Rightarrow f_1 = -\frac{1}{x-2}\left(-\frac{x^2}{2x-4}\right)' + \frac{c}{x-2}$$

$$= \frac{1}{x-2}\left(\frac{2x}{2x-4} - \frac{x^2(2)}{(2x-4)^2}\right) + \frac{c}{x-2}$$

$$= \frac{1}{x-2}\left(\frac{2x(2x-4)-2x^2}{(2x-4)^2}\right) + \frac{c}{x-2}$$

$$= \frac{1}{4}\,\frac{2x^2-8x}{(x-2)^3} + \frac{c}{x-2} = \frac{x^2-4x}{2(x-2)^3} + \frac{c}{x-2}$$

$f_1(0)=0 \Rightarrow \frac{c}{-2} = 0 \Rightarrow c=0$ so

$$f_1 = \frac{x(x-4)}{2(x-2)^3}$$

$$\rightarrow f^{outer}(x) = 2 - \frac{x^2}{2x-4} + \varepsilon\,\frac{x(x-4)}{2(x-2)^3} + \cdots$$

• inner solution.

let $s = \frac{x-1}{\varepsilon^p}$ then: $x = 1+\varepsilon^p s$

$$\varepsilon^{1-2p}\frac{d^2f}{ds^2} + \varepsilon^{-p}(1+\varepsilon^p s-2)\frac{df}{ds} + f = 2-1-\varepsilon^p s$$

$\Rightarrow$ requires $\varepsilon^{1-2p} = \varepsilon^{-p} \Rightarrow p=1$

so $$\frac{1}{\varepsilon}\frac{d^2f}{ds^2} + \left(-\frac{1}{\varepsilon}+s\right)\frac{df}{ds} + f = 1-\varepsilon s$$

let $f = f_0 + \varepsilon f_1 + \cdots$

$\Rightarrow$ to $O(\frac{1}{\varepsilon})$:

$$\frac{d^2f_0}{ds^2} - \frac{df_0}{ds} = 0 \quad \Rightarrow f_0(s) = A_0e^s + B_0$$

BC: at $x=1$, $s=0$ and so $f_0(s=0)=1 \Rightarrow$

$A_0+B_0 = 1 \Rightarrow B_0 = 1-A_0$

so $f_0(s) = A_0e^s + 1 - A_0$

$\Rightarrow$ to $O(1)$:

$$\frac{d^2f_1}{ds^2} - \frac{df_1}{ds} + s\frac{df_0}{ds} + f_0 = 1$$

6.

So $\frac{d^2 f_1}{ds^2} - \frac{df_1}{ds} = 1 - f_0 - s\frac{df_0}{ds}$

$$= 1 - A_0 e^s - 1 + A_0 - s\left(A_0 e^s\right)$$

$$= A_0 - A_0 e^s - sA_0 e^s$$

$\Rightarrow$ we need as solution:

$$f_1(s) = A_1 e^s + B_1 + \underbrace{C_1 s + C_2 s e^s + C_3 s^2 e^s}_{PS}.$$

$$\frac{df_1^{PS}}{ds} = C_1 + C_2 e^s + C_2 s e^s + 2sC_3 e^s + C_3 s^2 e^s$$

$$\frac{d^2 f_1^{PS}}{ds^2} = 2C_2 e^s + C_2 s e^s + 2C_3 e^s + 4sC_3 e^s + C_3 s^2 e^s$$

$$\Rightarrow \quad 2C_2 e^s + \cancel{C_2 s e^s} + 2C_3 e^s + 4sC_3 e^s + \cancel{C_3 s^2 e^s}$$

$$- C_1 - C_2 e^s - \cancel{C_2 s e^s} - 2sC_3 e^s - \cancel{C_3 s^2 e^s}$$

$$= A_0 - A_0 e^s - sA_0 e^s$$

$$\Rightarrow \begin{cases} -C_1 = A_0 & \Rightarrow C_1 = -A_0 \\ 2C_2 + 2C_3 - C_2 = -A_0 & \Rightarrow C_2 = -A_0 + A_0 = 0 \\ 4C_3 - 2C_3 = -A_0 & \Rightarrow C_3 = -\frac{A_0}{2} \end{cases}$$

$$\Rightarrow f_1(s) = A_1 e^s + B_1 - A_0 s - \frac{A_0}{2} s^2 e^s.$$

BC: $f_1(0) = 0 \Rightarrow A_1 + B_1 = 0 \Rightarrow B_1 = -A_1$ so

$$f_1(s) = A_1(e^s - 1) - A_0 s - \frac{A_0}{2} s^2 e^s$$

- Matching:
  - Taylor expand outer near $x = 1$:

$$f'_{outer}(x) = -\left(\frac{2x}{2x-4} - \frac{2x^2}{(2x-4)^2}\right) + \varepsilon\underbrace{[\ldots]}_{[f_1^{outer}(x)]'}.$$

$$= -\left(\frac{2x^2 - 8x}{(2x-4)^2}\right) + \varepsilon\left[f_1^{outer}(x)\right]'$$

so

$$f^{outer}(x) \simeq f_0^{outer}(1) + \left[f_0^{outer}(1)\right]' \cdot (x-1) + \varepsilon\left[f_1^{outer}(1) + \dots\right]$$

$$\simeq 2 - \frac{1}{(-2)} + (x-1)\left[\frac{6}{4}\right] + \varepsilon\left(\frac{-3}{2.(-1)}\right)$$

$$= \frac{5}{2} + \frac{3}{2}(x-1) + \frac{3}{2}\varepsilon + \dots$$

$$= \frac{5}{2} + \frac{3}{2}\varepsilon s + \frac{3}{2}\varepsilon + \dots$$

meanwhile, $\lim_{s\to-\infty} f^{inner}(s) = 1 - A_0 + \varepsilon\left[-A_1 - A_0 s\right]$

so : $\begin{cases} 1 - A_0 = \frac{5}{2} & \Rightarrow \quad A_0 = -\frac{3}{2} \\ \frac{3}{2} + \cancel{\frac{3}{2}s} = -A_1 - \cancel{A_0 s} & \Rightarrow \quad A_1 = -\frac{3}{2} \end{cases}$

so finally:

$$f^{inner}(s) = -\frac{3}{2}e^{s} + \frac{5}{2} + \varepsilon\left[-\frac{3}{2}(e^{s}-1) + \frac{3}{2}s + \frac{3}{4}s^2 e^{s}\right].$$

<u>composite exansion:</u>

$$f_{composite} = f^{outer}(x) + f^{inner}(x) - L$$

$$= 2 - \frac{x^2}{2x-4} + \varepsilon\frac{x(x-4)}{2(x-2)^3}$$
$$- \frac{3}{2}e^{\frac{x-1}{\varepsilon}} + \cancel{\frac{5}{2}} + \varepsilon\left[-\frac{3}{2}\left(e^{\frac{x-1}{\varepsilon}} - \cancel{1}\right) + \cancel{\frac{3}{2}\left(\frac{x-1}{\varepsilon}\right)} + \frac{3}{4}\left(\frac{x-1}{\varepsilon}\right)^2 e^{\frac{x-1}{\varepsilon}}\right]$$
$$- \cancel{\frac{5}{2}} - \cancel{\frac{3}{2}(x-1)} - \cancel{\frac{3}{2}\varepsilon}$$

$$= 2 - \frac{x^2}{2x-4} + \varepsilon\frac{x(x-4)}{2(x-2)^3}$$
$$- \frac{3}{2}e^{\frac{x-1}{\varepsilon}} + \varepsilon\left[\frac{3}{4}\left(\frac{x-1}{\varepsilon}\right)^2 e^{\frac{x-1}{\varepsilon}} - \frac{3}{2}e^{\frac{x-1}{\varepsilon}}\right].$$

as required.

Problem 2

(i) $\epsilon \dfrac{d^2 f}{dx^2} - \dfrac{df}{dx} + \dfrac{1}{f} = 0 \qquad \varepsilon > 0$

since velocity field is >0, b.l is on the right (at $x=1$)

• outer: let $f = f_0 + \epsilon f_1 + \dots$

to 0th order: $-\dfrac{df_0}{dx} + \dfrac{1}{f_0} = 0$

$\Rightarrow \quad df_0 \cdot f_0 = dx$

$\Rightarrow \quad \frac{1}{2} f_0^2 = x + C$

applying BC @ $x=0$ $f_0(0) = 2 \Rightarrow$

$\frac{1}{2} \cdot 4 = 0 + C \Rightarrow C = 2$ so

$f_0^2(x) = 2(x+2)$

$f_0^{outer}(x) = \pm\sqrt{2x+4} \rightarrow$ which to keep?

$f_0(0) = 2 \Rightarrow \oplus$ sol only.

• inner let $s = \dfrac{x-1}{\varepsilon^p}$ then:

$\varepsilon^{1-2p} \dfrac{d^2 f}{ds^2} - \varepsilon^{-p} \dfrac{df}{ds} + \dfrac{1}{f} = 0$

so we need $1 - 2p = -p \Rightarrow p = 1$

$\rightarrow$ to lowest order:

$\dfrac{d^2 f_0}{ds^2} - \dfrac{df_0}{ds} = 0 \Rightarrow f_0^{inner}(s) = A_0 e^s + B_0.$

Applying BC @ $x=1$ $(s=0)$: $f_0(0) = 1 \Rightarrow$

$A_0 + B_0 = 1$

$\Rightarrow B_0 = 1 - A_0$

so $f_0^{inner}(s) = A_0 e^s + 1 - A_0$

Matching:

$$\lim_{x\to 1} f_0^{outer}(x) = \lim_{s\to -\infty} f_0^{inner}(s)$$

$$\sqrt{6} = 1 - A_0 \Rightarrow A_0 = 1-\sqrt{6}$$

so $f_0^{inner}(s) = (1-\sqrt{6})e^{s} + \sqrt{6}$

Composite expansion:

$$f_{compo}(x) = \sqrt{2x+4} + (1-\sqrt{6})e^{\frac{x-1}{\varepsilon}} + \cancel{\sqrt{6}} - \cancel{\sqrt{6}}$$

$$= \sqrt{2x+4} + (1-\sqrt{6})e^{x-1}.$$

(iii) $\varepsilon \frac{d^2 f}{dx^2} + 2\frac{df}{dx} + x f^2 = 0 \qquad \varepsilon > 0.$

- velocity field $< 0 \rightarrow$ BL is on left $(x=0)$
- outer: $2\frac{df_0}{dx} + x f_0^2 = 0$

$$\Rightarrow -2\frac{df_0}{f_0^2} = x\,dx$$

$$\Rightarrow +\frac{2}{f_0} = \frac{x^2}{2} + C$$

at $x = 1$, $f_0(1) = 2 \Rightarrow$

$$+\frac{2}{2} = \frac{1}{2} + C \Rightarrow C = \frac{1}{2}$$

so $f_0^{outer} = 2 \cdot \frac{1}{\frac{x^2}{2}+\frac{1}{2}} = \frac{4}{x^2+1}$

- inner: let $s = \frac{x}{\varepsilon^p} \Rightarrow \varepsilon^{1-2p}\frac{d^2 f}{ds^2} + 2\varepsilon^{-p}\frac{df}{ds} + \varepsilon^p s f^2 = 0$

$$\Rightarrow \varepsilon^{1-2p} = \varepsilon^{-p} \Rightarrow p = 1 \text{ so } x = \varepsilon s.$$

To lowest order: $\frac{d^2 f_0}{ds^2} + 2\frac{df_0}{ds} = 0 \Rightarrow$

$$f_0^{inner}(s) = A_0 e^{-2s} + B_0$$

BC at $x=0$ $(s=0)$: $f_0^{inner}(0) = A_0 + B_0 = 0 \Rightarrow B_0 = -A_0$

$$\rightarrow \quad f_0^{inner}(s) = A_0 (e^{-2s} - 1)$$

- <u>Matching:</u>

$$\lim_{x \to 0} f^{outer}(x) = \lim_{s \to +\infty} f^{inner}(s)$$

$$4 = -A_0 \Rightarrow A_0 = -4 \text{ so}$$

$$f_0^{inner}(s) = 4(1 - e^{-2s})$$

- <u>Composite expansion:</u>

$$f_{compo}(x) = \frac{4}{x^2+1} + 4(\cancel{1} - e^{-\frac{2x}{\varepsilon}}) - \cancel{4}$$

$$= \frac{4}{x^2+1} - 4e^{-\frac{2x}{\varepsilon}}$$