HOMEWORK 5

Problem 1 $\epsilon f'' + f' + f = 0$ where $\epsilon < 0$

$f(0) = e$

$f(1) = 0$

If $\epsilon < 0$, lets write $\eta = -\epsilon$ so the new problem is

$$\begin{cases} \eta f'' - f' - f = 0 \\ f(0) = e \quad f(1) = 0 \end{cases}$$

$\epsilon > 0$

Note $V(x) > 0$ so BL is at $x = 1$

① Exact solution

Characteristic polynomial $\eta \lambda^2 - \lambda - 1 = 0$

$$\Rightarrow \lambda_{1,2} = \frac{1 \pm \sqrt{1 + 4\eta}}{2\eta}$$

Note: since η is small, the roots are both positive.

$\rightarrow \quad f(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$

$f(0) = e \Rightarrow A + B = e$

$f(1) = 0 \Rightarrow Ae^{\lambda_1} + Be^{\lambda_2} = 0$

$\Rightarrow B = -Ae^{\lambda_1 - \lambda_2}$

$$A - Ae^{\lambda_1 - \lambda_2} = e \Rightarrow A = \frac{e}{1 - e^{\lambda_1 - \lambda_2}} \qquad B = \frac{-e}{e^{\lambda_2 - \lambda_1} - }$$

Note: $\lambda_1 - \lambda_2 = \dfrac{\sqrt{1+4\eta}}{\eta}$

so

$$f(x) = \frac{e}{1 - e^{\sqrt{1+4\eta}/\eta}} \, e^{\left(\frac{1}{2\eta} + \frac{\sqrt{1+4\eta}}{2\eta}\right)x} + \frac{e}{1 - e^{-\sqrt{1+4\eta}/\eta}} \, e^{\left(\frac{1}{2\eta} - \frac{\sqrt{1+4\eta}}{2\eta}\right)x}$$

Plot of real solution confirms bl is at $x = 1$ & suggests thickness is $O(\eta)$

② Multi-scale analysis

let $X_0 = \dfrac{x-1}{\eta} \qquad X_1 = x$

$$\frac{d}{dx} = \frac{dX_0}{dx}\frac{\partial}{\partial X_0} + \frac{dX_1}{dx}\frac{\partial}{\partial X_1} = \eta^{-1}\frac{\partial}{\partial X_0} + \frac{\partial}{\partial X_1}$$

so $$\frac{d^2}{dx^2} = \eta^{-2}\frac{\partial^2}{\partial X_0^2} + 2\eta^{-1}\frac{\partial^2}{\partial X_0 \partial X_1} + \frac{\partial^2}{\partial X_1^2}$$

$$\Rightarrow \quad \frac{1}{\eta}\frac{\partial^2 f}{\partial X_0^2} + 2\frac{\partial^2 f}{\partial X_0 \partial X_1} + \eta\frac{\partial^2 f}{\partial X_1^2} - \frac{1}{\eta}\frac{\partial f}{\partial X_0} - \frac{\partial f}{\partial X_1} - f = 0$$

let $f = f_0 + \eta f_1 + \cdots$

$\Rightarrow$ To lowest order:

$$\frac{\partial^2 f_0}{\partial X_0^2} - \frac{\partial f_0}{\partial X_0} = 0 \quad \Rightarrow \quad f_0(X_0, X_1) = A(X_1)e^{+X_0} + B(X_1)$$

To next order

$$2\frac{\partial^2 f_0}{\partial X_0 \partial X_1} + \frac{\partial^2 f_1}{\partial X_0^2} - \frac{\partial f_0}{\partial X_1} - \frac{\partial f_1}{\partial X_0} - f_0 = 0$$

$$\Rightarrow \quad \frac{\partial^2 f_1}{\partial^2 X_0^2} - \frac{\partial f_1}{\partial X_0} = -2\frac{\partial^2 f_0}{\partial X_0 \partial X_1} + \frac{\partial f_0}{\partial X_1} + f_0$$

$$= -2\left[\frac{dA}{dX_1}e^{X_0}\right] + \frac{dA}{dX_1}e^{X_0} + \frac{dB}{dX_1} + A(X_1)e^{X_0} + B$$

To eliminate all secular terms we need

$$-\frac{dA}{dX_1} + A(X_1) = 0 \quad \Rightarrow \quad A(X_1) = Ce^{X_1}$$

$$\frac{dB}{dX_1} + B = 0 \quad \Rightarrow \quad B(X_1) = De^{-X_1}$$

At $x = 0$, $f(0) = e \Rightarrow$ at $X_0 = -\frac{1}{\eta}$, $X_1 = 0$, $f(-\frac{1}{\eta}, 0) = e$

this implies

$A(0)e^{-\frac{1}{\eta}} + B(0) = e$

At $x=1$ $f(1)=0$ $\Rightarrow$ at $X_0=0,\ X_1=1$ $f(0,1)=0$

$$\Rightarrow \quad A(1)+B(1)=0$$

$$\Rightarrow \begin{cases} Ce^{-\frac{1}{\eta}}+D = e \\ Ce + \frac{D}{e} = 0 \end{cases} \quad \begin{matrix} \Rightarrow & Ce^{-\frac{1}{\eta}} - Ce^2 = e \\ \Rightarrow & D = -Ce^2 \end{matrix}$$

$$C = \frac{e}{e^{-\frac{1}{\eta}}-e^2} = \frac{1}{e^{-1-\frac{1}{\eta}}-e}$$

so $$f_0(X_0,X_1) = \frac{1}{e^{-1-\frac{1}{\eta}}-e}e^{X_1+X_0} - \frac{e^2}{e^{-1-\frac{1}{\eta}}-e}e^{-X_1}$$

$$\Rightarrow f(x) = \frac{1}{e^{-1-\frac{1}{\eta}}-e}\left(e^{\frac{x-1}{\eta}+x} - e^{2-x}\right) \simeq \frac{e^{2-x} - e^{\frac{x-1}{\eta}}}{e}$$

$$\simeq e^{1-x} - e^{\frac{x-1}{\eta}}$$ ignoring 1 in front of $\frac{1}{\eta}$

Comparing this with true solution for small η:

$$\frac{1}{2\eta} + \frac{\sqrt{1+4\eta}}{2\eta} \simeq \frac{1}{2\eta} + \frac{1+\frac{4\eta}{2}}{2\eta} \cong \frac{1}{\eta}+1$$

$$\frac{1}{2\eta} - \frac{\sqrt{1+4\eta}}{2\eta} \simeq \frac{1}{2\eta} - \frac{1+\frac{4\eta}{2}}{2\eta} \cong -1+O(\eta)$$

so $$f(x) \simeq \frac{e}{1-e^{\frac{1}{\eta}+2}}e^{(\frac{1}{\eta}+1)x} + \frac{e}{1-e^{-\frac{1}{\eta}-2}}e^{-x}$$

$$\simeq \frac{e^{1+\frac{x}{\eta}+x}}{-e^{\frac{1}{\eta}}} + \frac{e^{1-x}}{1}$$

$$\simeq -e^{\frac{x-1}{\eta}} + e^{1-x}$$ ignoring $1+x$ in front of $\frac{x}{\eta}$

$\rightarrow$ They match!

③ Boundary layer analysis

Outer solution (x not close to 1): let $f = f_0 + \eta f_1 + \cdots$

$$\Rightarrow \begin{cases} f_0' + f_0 = 0 \\ f_0(0) = e \end{cases} \rightarrow \quad f_0(x) = Ce^{-x}$$

$e = C$ so

$$f_0^{outer}(x) = e^{1-x}$$

Inner solution: $\quad s = \frac{x-1}{\eta}$ then

$$\frac{d}{dx} = \frac{1}{\eta}\frac{d}{ds} \qquad \frac{d^2}{dx^2} = \frac{1}{\eta^2}\frac{d^2}{ds^2}$$

$\rightarrow$ to lowest order $\quad \frac{d^2 f_0}{ds^2} - \frac{df_0}{ds} = 0$

So $\quad f_0(s) = Ae^{s} + B$

Near $x=1$, $f_0(0) = 0 \quad \rightarrow \quad A + B = 0$ so $B = -A$.

So $f_0^{inner}(s) = A(e^s - 1)$

Prandtl's matching condition:

$$\lim_{x \to 1} f_0^{outer}(x) = \lim_{s \to -\infty} f^{inner}(s)$$

$$1 = -A \quad \Rightarrow A = -1$$

So $\quad f_0^{inner}(s) = 1 - e^{s}$

$\rightarrow$ finally

$$f_{compo}(x) = f_0^{outer}(x) + f_0^{inner}(x) - L$$

$$= e^{1-x} + 1 - e^{\frac{x-1}{\eta}} - 1 = e^{1-x} - e^{\frac{x-1}{\eta}}$$

$\rightarrow$ as for the limit of the exact solution as $\eta \to 0$.

Problem 2

$$\begin{cases} \varepsilon \dfrac{d^2 f}{dx^2} + \dfrac{df}{dx} + e^{f} = 0 \\ f(0) = 1 \qquad f(1) = -\ln 2 \end{cases}$$

① Boundary layer analysis

The velocity field $v(x) < 0 \Rightarrow$ BL is on the left @ $x=0$

- Let's look at outer (not too close to 0)

let $f = f_0 + \varepsilon f_1 + \cdots$

$\rightarrow$ to lowest order $\dfrac{df_0}{dx} + e^{f_0} = 0$

Separation of variables:

$$\frac{df_0}{e^{f_0}} = -dx \Rightarrow e^{-f_0}\, df_0 = -dx$$

$$\Rightarrow -e^{-f_0} = -x + C$$

At $x=1$, $f(1) = -\ln 2$ so

$$-e^{-(-\ln 2)} = -1 + C$$

$$-2 = C - 1 \Rightarrow C = -1 \text{ so}$$

$$e^{-f_0} = 1 + x \Rightarrow f_0^{outer}(x) = -\ln(1+x)$$

- Near the inner boundary (near $x=0$) let $s = \dfrac{x}{\varepsilon}$ then to lowest order

$$\frac{1}{\varepsilon}\frac{d^2 f_0}{ds^2} + \frac{1}{\varepsilon}\frac{df_0}{ds} + e^{f_0 + \cdots} = 0$$

$$\Rightarrow \frac{d^2 f_0}{ds^2} + \frac{df_0}{ds} = 0$$

$$\Rightarrow f_0^{inner}(s) = Ae^{-s} + B$$

$f_0^{inner}(0) = 1 \Rightarrow A + B = 1 \Rightarrow B = 1 - A$ so

$$f_0^{inner}(s) = Ae^{-s} + 1 - A = A(e^{-s} - 1) + 1$$

- Matching:

$$\lim_{x \to 0} f_0^{outer} = \lim_{s \to +\infty} f^{inner}(s)$$

$$-\ln(1) = -A + 1 \Rightarrow A = 1 \quad \text{so}$$

$$f_0^{inner}(s) = e^{-s}$$

- Composite expansion:

$$f_{compo}(x) = f_0^{outer}(x) + f_0^{inner}(x) - L$$

$$= -\ln(1+x) + e^{-\frac{x}{\varepsilon}} - 0 \qquad \text{as required}$$

② Note on the problem of the multiscale expansion.

Suppose we had tried, as usual, to let

$$X_0 = \frac{x}{\varepsilon} \qquad X_1 = x$$

then the eq becomes

$$\frac{1}{\varepsilon}\frac{\partial^2 f}{\partial X_0^2} + 2\frac{\partial^2 f}{\partial X_0 \partial X_1} + \varepsilon\frac{\partial^2 f}{\partial X_1^2} + \frac{1}{\varepsilon}\frac{\partial f}{\partial X_0} + \frac{\partial f}{\partial X_1} + e^f = 0$$

→ to lowest order; assuming $f = f_0 + \varepsilon f_1 + \cdots$

$$\frac{\partial^2 f_0}{\partial X_0^2} + \frac{\partial f_0}{\partial X_0} = 0 \Rightarrow f_0(X_0, X_1) = A(X_1)e^{-X_0} + B(X_1)$$

→ next order:

$$\frac{\partial^2 f_1}{\partial X_0^2} + 2\frac{\partial^2 f_0}{\partial X_0 \partial X_1} + \frac{\partial f_1}{\partial X_0} + \frac{\partial f_0}{\partial X_1} + e^{f_0} = 0.$$

$\rightarrow$

$$\frac{\partial^2 f_1}{\partial X_0^2} + \frac{\partial f_1}{\partial X_0} = -2\frac{\partial^2 f_0}{\partial X_0 \partial X_1} - \frac{\partial f_0}{\partial X_1} - e^{f_0}$$

$$= -2\left[-\frac{dA}{dX_1}e^{-X_0}\right] - \left[\frac{dA}{dX_1}e^{-X_0} + \frac{dB}{dX_1}\right] - e^{Ae^{-X_0}+B}$$

$$= \frac{dA}{dX_1}e^{-X_0} - \frac{dB}{dX_1} - \underbrace{e^{Ae^{-X_0}+B}}$$

Problem: we don't really know what to do with this!

If we suppose that this term does not affect secular terms & just ignore it, we set $\frac{dA}{dX_1} = \frac{dB}{dX_1} = 0 \Rightarrow A(X_1) = C \quad , \quad B(X_1) = D$

Applying BCs, we get :

$$f(0) = 1 \Rightarrow f(0,0) = 1 \Rightarrow A(0) + B(0) = 1$$

$$f(1) = -\ln 2 \Rightarrow f(\tfrac{1}{\varepsilon}, 1) = -\ln 2$$

$$\Rightarrow A(1)e^{-\frac{1}{\varepsilon}} + B(1) = -\ln 2$$

$$\rightarrow \begin{cases} C + D = 1 & \Rightarrow D = 1 - C \\ Ce^{-\frac{1}{\varepsilon}} + D = -\ln 2 & \Rightarrow Ce^{-\frac{1}{\varepsilon}} + 1 - C = -\ln 2 \end{cases}$$

$$\Rightarrow C = \frac{-\ln 2 - 1}{e^{-1/\varepsilon} - 1}$$

And then

$$f(X_0, X_1) = \frac{1 + \ln 2}{1 - e^{-1/\varepsilon}} e^{-\frac{x}{\varepsilon}} + 1 - \frac{\ln 2 + 1}{e^{-\frac{1}{\varepsilon}} - 1}.$$

$\rightarrow$ not at all like the BL analysis!

$\Rightarrow$ The method of multiple scale fails here.