HOMEWORK 2

Problem 1 $e^{-x} - x - 1 + \epsilon = 0$

Actually, the iterative method is not informative here. My apologies.

Let's try $x = a_0 + a_1\epsilon + a_2\epsilon^2 + \dots$

$$e^{-(a_0 + a_1\epsilon + a_2\epsilon^2 + \dots)} - (a_0 + a_1\epsilon + a_2\epsilon^2 + \dots) - 1 + \epsilon = 0$$

Since $e^{-x} \sim 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$ then this becomes

$$1 - (a_0 + a_1\epsilon + a_2\epsilon^2 + \dots) + \frac{(a_0 + a_1\epsilon + a_2\epsilon^2 + \dots)^2}{2} - \frac{(a_0 + a_1\epsilon + a_2\epsilon^2 + \dots)^3}{6}$$

$$- (a_0 + a_1\epsilon + a_2\epsilon^2 + \dots) - 1 + \epsilon = 0$$

to 0th order: $\underbrace{1 - a_0 + \frac{a_0^2}{2} - \frac{a_0^3}{6} + \dots}_{e^{-a_0}} \; a_0 - 1 = 0$

$e^{-a_0} + a_0 - 1 = 0 \rightarrow$ trivial solution of $a_0 = 0$

$\Rightarrow$ in fact $x = a_1\epsilon + a_2\epsilon^2 + a_3\epsilon^3 + \dots$

so $$\cancel{1} - (a_1\epsilon + a_2\epsilon^2 + a_3\epsilon^3 + \dots) + \frac{(a_1\epsilon + a_2\epsilon^2 + a_3\epsilon^3 + \dots)^2}{2}$$

$$- \frac{(a_1\epsilon + a_2\epsilon^2 + a_3\epsilon^3 + \dots)^3}{6} - (a_1\epsilon + a_2\epsilon^2 + a_3\epsilon^3 + \dots) + \epsilon - \cancel{1} = 0$$

$\Rightarrow$ to first order only:

$$-a_1 - a_1 + 1 = 0 \Rightarrow \boxed{a_1 = \frac{1}{2}}$$

$\Rightarrow$ to second order only:

$$-a_2 + \frac{a_1^2}{2} - a_2 = 0 \Rightarrow \boxed{a_2 = \frac{a_1^2}{4} = \frac{1}{16}}$$

$\Rightarrow$ to 3rd order only :

$$-a_3 + a_1a_2 - \frac{a_1^3}{6} - a_3 = 0$$

$$\Rightarrow \quad a_3 = \frac{1}{2}\left(a_1a_2 - \frac{a_1^3}{6}\right) = \frac{1}{2}\left(\frac{1}{32} - \frac{1}{48}\right) = \boxed{\frac{1}{192} = a_3}$$

$\Rightarrow$ to 4th order only

$$-a_4 + a_1a_3 + \frac{1}{2}a_2^2 - \frac{3a_1^2a_2}{6} - a_4 = 0$$

$$a_4 = \frac{1}{2}\left(a_1a_3 + \frac{1}{2}a_2^2 - \frac{3a_1^2a_2}{6}\right)$$

$$\boxed{a_4 = -\frac{5}{3072}}$$

So finally,

$$x = \frac{\epsilon}{2} + \frac{\epsilon^2}{16} + \frac{\epsilon^3}{192} - \frac{5\epsilon^4}{3072} + o(\epsilon^5)$$

Check: Using the NR routine from AMS213 :

$\epsilon = 0.1$: actual solution $= 0.050630174\ldots$
approximate solution $= 0.050630044\ldots$ ✓

Problem 2. Show that $Ci(x) = \int_x^\infty \frac{\cos t}{t}\,dt$

has the expansion $\left(\frac{1}{x^2} - \frac{3!}{x^4} + \ldots\right)\cos x - \left(\frac{1}{x} - \frac{2!}{x^3} + \ldots\right)$si

$\rightarrow$ This looks like an expansion obtained from an integration by parts.

$$Ci(x) = \int_x^\infty \frac{\cos t}{t}\,dt$$

← integrate this one (cos t)

← take derivative of this one (t) to get $\frac{1}{t}, \frac{1}{t^2}, \frac{1}{t^3}$, etc...

$$= \left[\frac{\sin t}{t}\right]_x^\infty + \int_x^\infty \frac{\sin t}{t^2}\,dt$$

$$= 0 - \frac{\sin x}{x} + \left[\frac{-\cos t}{t^2}\right]_x^\infty - 2\int_x^\infty \frac{\cos t}{t^3}\,dt$$

$$= -\frac{\sin x}{x} + \frac{\cos x}{x^2} - 2\left\{\left[\frac{\sin t}{t^3}\right] + 3\int_x^\infty \frac{\sin t}{t^4}\,dt\right\}$$

$$= -\frac{\sin x}{x} + \frac{\cos x}{x^2} + \frac{2\sin x}{x^3} + 6\left[\frac{\cos t}{t^4}\right]_x^\infty + \cdots$$

$$= \left(\frac{1}{x^2} - \frac{6}{x^4}\right)\cos x - \left(\frac{1}{x} - \frac{2}{x^3}\right)\sin x + \cdots$$ as required.

Problem 3

(i) $$\frac{1}{1+\varepsilon+2\varepsilon^2} \simeq 1 - (\varepsilon+2\varepsilon^2) + (\varepsilon+2\varepsilon^2)^2 - (\varepsilon+2\varepsilon^2)^3 + \cdots$$

$$\simeq 1 - \varepsilon - 2\varepsilon^2 + \varepsilon^2 + o(\varepsilon^3)$$

$$\simeq 1 - \varepsilon - \varepsilon^2 + o(\varepsilon^3)$$

(ii) $$\sqrt{1-\varepsilon^3+\varepsilon^6} = \sqrt{1-(\varepsilon^3-\varepsilon^6)}$$

$$\simeq 1 - \frac{1}{2}(\varepsilon^3-\varepsilon^6) - \frac{1}{4}(\varepsilon^3-\varepsilon^6)^2\cdot\frac{1}{2} + o(h.o.t)$$

$$\simeq 1 - \frac{\varepsilon^3}{2} + \frac{\varepsilon^6}{2} - \frac{\varepsilon^6}{8} + o(\varepsilon^9)$$

$$\simeq 1 - \frac{\varepsilon^3}{2} + \frac{3\varepsilon^6}{8} + o(\varepsilon^9)$$

(iii) $$\cos(\varepsilon-\varepsilon^2) \simeq 1 - \frac{(\varepsilon-\varepsilon^2)^2}{2} + \frac{(\varepsilon-\varepsilon^2)^4}{4!} + \cdots$$

$$\simeq 1 - \frac{\varepsilon^2}{2} + \varepsilon^3 + o(\varepsilon^4)$$

(iv) $$\ln(1+\sin\varepsilon) \simeq \ln\left(1+\varepsilon - \frac{\varepsilon^3}{6} + \frac{\varepsilon^5}{5!} + \cdots\right)$$

$$\simeq \left(\varepsilon - \frac{\varepsilon^3}{6} + \frac{\varepsilon^5}{5!} + \cdots\right) - \frac{1}{2}\left(\varepsilon - \frac{\varepsilon^3}{6} + \frac{\varepsilon^5}{5!} + \cdots\right)^2 +$$

$$\frac{1}{3}\left(\varepsilon - \frac{\varepsilon^3}{6} + \frac{\varepsilon^5}{5!} + \cdots\right)^3$$

$$\simeq \varepsilon - \frac{\varepsilon^2}{2} - \frac{\varepsilon^3}{6} + \frac{\varepsilon^3}{3} + o(\varepsilon^4)$$

$$\simeq \varepsilon - \frac{\varepsilon^2}{2} + \frac{\varepsilon^3}{6} + o(\varepsilon^4)$$

(v) $$\ln(1+e^{\varepsilon}) \simeq \ln\left(1+1+\varepsilon+\frac{\varepsilon^2}{2}+\frac{\varepsilon^3}{6}+\cdots\right)$$

$$\simeq \ln\left(2+\varepsilon+\frac{\varepsilon^2}{2}+\frac{\varepsilon^3}{6}+\cdots\right)$$

$$\simeq \ln\left[2\left(1+\frac{\varepsilon}{2}+\frac{\varepsilon^2}{4}+\frac{\varepsilon^3}{12}+\cdots\right]\right.$$

$$= \ln 2 + \ln\left(1+\frac{\varepsilon}{2}+\frac{\varepsilon^2}{4}+\cdots\right)$$

$$= \ln 2 + \left(\frac{\varepsilon}{2}+\frac{\varepsilon^2}{4}+\frac{\varepsilon^3}{12}+\cdots\right) - \frac{\left(\frac{\varepsilon}{2}+\frac{\varepsilon^2}{4}+\frac{\varepsilon^3}{12}\cdots\right)^2}{2}$$

$$= \ln 2 + \frac{\varepsilon}{2} + \frac{\varepsilon^2}{4} - \frac{\varepsilon^2}{8} + o(\varepsilon^3)$$

$$= \ln 2 + \frac{\varepsilon}{2} + \frac{\varepsilon^2}{8} + o(\varepsilon^3)$$

(vi) $$e^{1-\cos\epsilon} = e^{1-\left(1-\frac{\epsilon^2}{2}+\frac{\epsilon^4}{4!}+\cdots\right)} = e^{\frac{\varepsilon^2}{2}-\frac{\epsilon^4}{4!}+\frac{\epsilon^6}{6!}+\cdots}$$

$$= 1+\left(\frac{\varepsilon^2}{2}-\frac{\varepsilon^4}{4!}+\frac{\varepsilon^6}{6!}\cdots\right) + \frac{\left(\frac{\varepsilon^2}{2}-\frac{\varepsilon^4}{4!}+\frac{\varepsilon^6}{6!}\cdots\right)^2}{2} + \cdots$$

$$= 1+\frac{\varepsilon^2}{2}-\frac{\varepsilon^4}{4!}+\frac{\varepsilon^4}{8}+\cdots$$

$$= 1+\frac{\varepsilon^2}{2}+\frac{\varepsilon^4}{12}+o(\varepsilon^6)$$

Problem 4

$$\begin{cases} \frac{df}{dx} - \epsilon f + 1 = 0 \\ f(0) = \varepsilon \end{cases}$$

let $f = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \dots$

→ to 0th order:

$$\begin{cases} \frac{df_0}{dx} + 1 = 0 \\ f_0(0) = 0 \end{cases} \qquad f_0(x) = -x + K \quad \rightarrow K = 0 \quad \text{so} \quad f_0(x) = -x$$

→ to 1st order

$$\begin{cases} \frac{df_1}{dx} - f_0 = 0 \\ f_1(0) = 1 \end{cases} \qquad f_1(x) = -\frac{x^2}{2} + K \quad \longrightarrow K = 1$$

So $f_1(x) = 1 - \frac{x^2}{2}$

so $f(x) = -x + \varepsilon\left(1 - \frac{x^2}{2}\right) + \dots$ ← this expansion is non uniform since we cannot bound $\left|1 - \frac{x^2}{2}\right|$.

Exact solution:

$$f(x) = Ke^{\epsilon x} + \frac{1}{\epsilon}$$

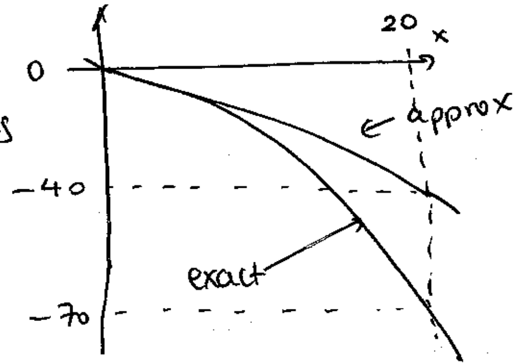
(general solution + particular integral).

To satisfy $f(0) = \epsilon$ we need

$$\epsilon = K + \frac{1}{\epsilon} \rightarrow K = \epsilon - \frac{1}{\epsilon} \text{ so}$$

$$f(x) = \left(\epsilon - \frac{1}{\epsilon}\right)e^{\epsilon x} + \frac{1}{\epsilon}$$

Solution looks like



Problem 5

$$\begin{cases} \dfrac{d^2f}{dx^2} - 2\dfrac{df}{dx} + \varepsilon f = 0 \\ f(0) = 0 \quad f(1) = 1 \end{cases}$$

let $f = f_0 + \varepsilon f_1 + \cdots$

- Oth. $\rightarrow \begin{cases} f_0'' - 2f_0' = 0 \\ f_0(0) = 0 \quad f_0(1) = 1 \end{cases}$

$\rightarrow f_0'(x) = Ke^{+2x}$

$\rightarrow f_0(x) = \dfrac{K}{2} e^{2x} + K'$

to satisfy BCs:

$\dfrac{K}{2} + K' = 0$ and $\dfrac{K}{2} e^2 + K' = 1$

so $K' = -\dfrac{K}{2}$ $\quad \dfrac{K}{2} e^2 - \dfrac{K}{2} = 1$

$\Rightarrow K = \dfrac{2}{e^2-1} \quad K' = -\dfrac{1}{e^2-1}$

so $f_0(x) = \dfrac{1}{e^2-1}\left[e^{2x} - 1\right]$

- To first order:

$$\begin{cases} f_1'' - 2f_1' + f_0 = 0 \\ f_1(0) = f_1(1) = 0 \end{cases}$$

$$f_1'' - 2f_1' = \frac{-1}{e^2-1}\left(e^{2x} - 1\right) = -\frac{e^{2x}}{e^2-1} + \frac{1}{e^2-1}$$

The particular integral for the term in $\dfrac{1}{e^2-1}$ is simply such that $f_1'' = 0$ and $f_1' = \dfrac{-1}{2(e^2-1)} \Rightarrow f_1 = \dfrac{-x}{2(e^2-1)}$

The one for the term in $\dfrac{e^{2x}}{e^2-1}$ must be of the kind

$f_1^{ps} = xe^{2x}.K'' \Rightarrow K''$ satisfies

$$4e^{2x}K'' + \cancel{4xe^{2x}K''} - 2\left(K''e^{2x} + \cancel{2xe^{2x}K''}\right) = \frac{-e^{2x}}{e^2-1}.$$

$$K'' = \frac{-1}{2(e^2-1)}$$

So finally,

$$f_1(x) = Ae^{2x} + B - \frac{1}{2(e^2-1)}xe^{2x} - \frac{x}{2(e^2-1)}$$

to satisfy $f_1(0)=0 \Rightarrow A + B = 0 \Rightarrow B = -A$

$$f_1(1) = 0 \Rightarrow Ae^2 + B - \frac{e^2}{2(e^2-1)} - \frac{1}{2(e^2-1)} = 0$$

$$\Rightarrow A(e^2-1) - \frac{1}{2(e^2-1)}(e^2+1) = 0$$

$$\Rightarrow A = + \frac{e^2+1}{2(e^2-1)^2}$$

So $$f_1(x) = \frac{-x}{2(e^2-1)}\left(e^{2x}+1\right) + \frac{e^2+1}{2(e^2-1)^2}\left(e^{2x}-1\right)$$

and finally

$$f(x) = \frac{1}{e^2-1}\left\{ e^{2x}-1 + \varepsilon\left[\frac{e^2+1}{2(e^2-1)}(e^{2x}-1) - \frac{x}{2}(e^{2x}+1)\right]\right\}$$

$$= \frac{e^{2x}}{e^2-1}\left\{1-e^{-2x} + \varepsilon\left[\frac{e^2+1}{2(e^2-1)}\left(1-e^{-2x}\right) - \frac{x}{2}\left(1+e^{-2x}\right)\right]\right\}$$

↑ this term makes the expansion non uniform (cannot be bounded as $x \to \pm\infty$).

Exact solution: let $f(x) \propto e^{\lambda x}$ then

$$\lambda^2 - 2\lambda + \varepsilon = 0 \qquad \lambda = \frac{2 \pm \sqrt{4-4\varepsilon}}{2} = 1 \pm \sqrt{1-\varepsilon}$$

$$\Rightarrow f(x) = Ae^{(1+\sqrt{1-\varepsilon})x} + Be^{(1-\sqrt{1-\varepsilon})x}$$

$f(0)=0 \Rightarrow A + B = 0$

$f(1)=1 \Rightarrow Ae^{1+\sqrt{1-\varepsilon}} + Be^{1-\sqrt{1-\varepsilon}} = 1$

So $B = -A$ and

$$A\left(e^{1+\sqrt{1-\varepsilon}} - e^{1-\sqrt{1-\varepsilon}}\right) = 1 \qquad 2Ae\sinh\left(\sqrt{1-\varepsilon}\right) = 1$$

$$\Rightarrow \quad A = \frac{1}{2e\sinh\left(\sqrt{1-\varepsilon}\right)}$$

and finally,

$$f(x) = \frac{1}{2e\sinh\left(\sqrt{1-\varepsilon}\right)}\left(e^{(1+\sqrt{1-\varepsilon})x} - e^{(1-\sqrt{1-\varepsilon})x}\right)$$

$$f(x) = \frac{\left(e^{x-1}\right)\sinh\left(\sqrt{1-\varepsilon}\,x\right)}{\sinh\left(\sqrt{1-\varepsilon}\right)}$$

(Comparison between two solutions: plot it yourselves!)