SOLUTIONS FOR HW2

Exercise p 56

let $\delta_n(\varepsilon) = \sin^n(\varepsilon)$

(1) $\ln(1+\varepsilon)$.

Method # 1:

Say $\ln(1+\varepsilon) = a_0\delta_0(\varepsilon) + a_1\delta_1(\varepsilon) + a_2\delta_2(\varepsilon) + a_3\delta_3(\varepsilon) + \dots$

and expand in ε, then identify $a_0, a_1, a_2, a_3 \dots$

$$\ln(1+\varepsilon) = \varepsilon - \frac{\varepsilon^2}{2} + \frac{\varepsilon^3}{3} - \frac{\varepsilon^4}{4} \dots$$

$$(\sin\varepsilon)^n = \left(\varepsilon - \frac{\varepsilon^3}{3!} + \frac{\varepsilon^5}{5!} - \dots\right)^n$$

$$= \varepsilon^n\left(1 - \frac{\varepsilon^2}{6} + \frac{\varepsilon^4}{20} \dots\right)^n$$

$$= \varepsilon^n\left(1 - n\frac{\varepsilon^2}{6} + n\frac{\varepsilon^4}{20} \dots + \frac{n(n-1)}{2}\left(-\frac{\varepsilon^2}{6} + \frac{\varepsilon^4}{20}\dots\right)^2 + \dots\right)$$

$$= \varepsilon^n\left(1 - n\frac{\varepsilon^2}{6} + O(\varepsilon^4)\right)$$

$\rightarrow$

$$\sin\varepsilon = \varepsilon\left(1 - \frac{\varepsilon^2}{6} + O(\varepsilon^4)\right)$$

$$\sin^2\varepsilon = \varepsilon^2\left(1 - \frac{\varepsilon^2}{3} + O(\varepsilon^4)\right)$$

$$\sin^3\varepsilon = \varepsilon^3\left(1 - \frac{\varepsilon^2}{2} + O(\varepsilon^4)\right)$$

So finally:

$$\varepsilon - \frac{\varepsilon^2}{2} + \frac{\varepsilon^3}{3} - \frac{\varepsilon^4}{4} \cdots = a_0 + a_1\left(\varepsilon - \frac{\varepsilon^3}{6} + \cdots\right)$$
$$+ a_2\left(\varepsilon^2 - \frac{\varepsilon^5}{3} + \cdots\right)$$
$$+ a_3\left(\varepsilon^3 - \frac{\varepsilon^5}{2} \cdots\right)$$

$$\Rightarrow \begin{cases} a_0 = 0 \\ a_1 = 1 \\ a_2 = -\frac{1}{2} \\ -\frac{a_1}{6} + a_3 = \frac{1}{3} \Rightarrow a_3 = \frac{1}{3} + \frac{1}{6} = \frac{1}{2} \end{cases}$$

So $\ln(1+\varepsilon) = \sin\varepsilon - \frac{1}{2}\sin^2\varepsilon + \frac{1}{6}\sin^3\varepsilon + O(\sin^4\varepsilon)$

(2) e^{ε}

Method # 2 (See lecture notes)

$$a_0 = \lim_{\varepsilon\to 0} \frac{e^{\varepsilon}}{\delta_0(\varepsilon)} = \lim_{\varepsilon\to 0} \frac{e^{\varepsilon}}{1} = 1$$

$$a_1 = \lim_{\varepsilon\to 0} \frac{e^{\varepsilon} - a_0\delta_0(\varepsilon)}{\delta_1(\varepsilon)} = \lim_{\varepsilon\to 0} \frac{e^{\varepsilon} - 1}{\sin\varepsilon}$$ ← use l'Hopital

$$= \lim_{\varepsilon\to 0} \frac{e^{\varepsilon}}{\cos\varepsilon} = 1$$

$$a_2 = \lim_{\varepsilon\to 0} \frac{e^{\varepsilon} - a_0\delta_0(\varepsilon) - a_1\delta_1(\varepsilon)}{\sin^2\varepsilon} = \lim_{\varepsilon\to 0} \frac{e^{\varepsilon} - 1 - \sin\varepsilon}{\sin^2\varepsilon}$$

$$= \lim_{\varepsilon\to 0} \frac{e^{\varepsilon} - \cos\varepsilon}{2\sin\varepsilon\cos\varepsilon} = \lim_{\varepsilon\to 0} \frac{e^{\varepsilon} + \sin\varepsilon}{2\cos 2\varepsilon} = \frac{1}{2}$$

this is $\sin 2\varepsilon$

so $$e^{\varepsilon} = 1 + \sin\varepsilon + \frac{1}{2}\sin^2\varepsilon + O(\sin^3\varepsilon)$$

Exercise 1 p 61

Region of non-uniformity:

Calculate $\left|\frac{R_{n+1}}{R_n}\right|$, & see when it is larger than $O(1)$

(1) $$1 + \frac{\varepsilon x^2}{2!} + \frac{\varepsilon^2 x^4}{4!} + \cdots + \frac{\varepsilon^n x^{2n}}{(2n)!} + \cdots$$

$$\left|\frac{R_{n+1}}{R_n}\right| = \frac{\varepsilon^{n+1} x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{\varepsilon^n x^{2n}}$$

$$= \frac{\varepsilon x^2}{(2n+2)(2n+1)}$$

$\rightarrow$ this is $O(1)$ when $x^2 = O\left(\frac{1}{\varepsilon}\right)$

$\Leftrightarrow$ $x = O\left(\frac{1}{\sqrt{\varepsilon}}\right)$

(2) $$1 + \frac{\varepsilon^2}{2x} + \frac{\varepsilon^4}{3x^2} + \cdots \frac{\varepsilon^{2n}}{nx^n} + \cdots$$

$$\left|\frac{R_{n+1}}{R_n}\right| = \frac{\varepsilon^{2n+2}}{(n+1)x^{n+1}} \cdot \frac{nx^n}{\varepsilon^{2n}}$$

$$= \frac{\varepsilon^2}{x}\frac{n}{n+1}$$

$\rightarrow$ this is $O(1)$ when $x = O(\varepsilon^2)$

(3) $\sqrt{x} - \varepsilon + \frac{\varepsilon^2}{\sqrt{x}} - \frac{\varepsilon^3}{x} + \cdots + (-1)^n \varepsilon^n x^{\frac{1-n}{2}} + \cdots$

$$\left|\frac{R_{n+1}}{R_n}\right| = \left|\frac{(-1)^{n+1}\varepsilon^{n+1}x^{\frac{1-(n+1)}{2}}}{(-1)^n\varepsilon^n x^{\frac{1-n}{2}}}\right|$$

$$= \varepsilon x^{-\frac{1}{2}}$$

This is $O(1)$ when $x^{-\frac{1}{2}} = O\left(\frac{1}{\varepsilon}\right)$

$x = O(\varepsilon^2)$

(4) $1 - \varepsilon e^{-x} + \varepsilon^2 e^{-2x} + \varepsilon^3 e^{-3x} + \cdots + \varepsilon^n e^{-nx} + \cdots$

$$\left|\frac{R_{n+1}}{R_n}\right| = \frac{\varepsilon^{n+1} e^{-(n+1)x}}{\varepsilon^n e^{-nx}} = \varepsilon e^{-x}$$

$\rightarrow$ this is $O(1)$ when $e^x = O(\varepsilon)$

$x = O(\ln\varepsilon)$