CHAPTER 4: Multiple Scales

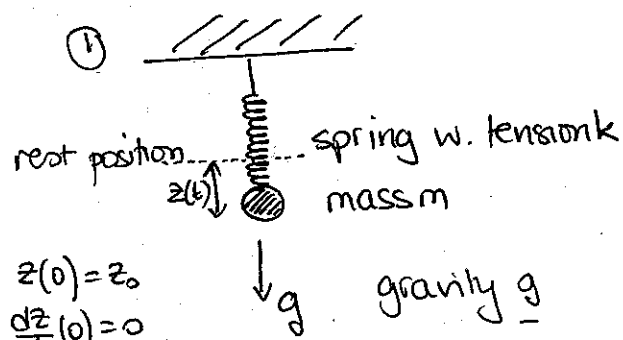
In the previous Chapter, we saw that there are cases of IVPs for which the method of strained coordinates does not work – as for example in the Van der Pol Oscillator.

To gain a better understanding of <u>why</u> this method doesn't work & how to fix the problem in this case let's look at an even simpler oscillator which exhibits the same problem.

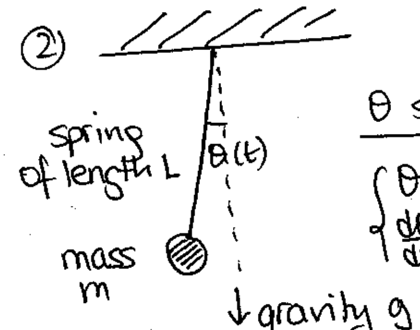
I. Introduction to multiple-scale problems

(A) The damped oscillator/pendulum

Consider the two mechanical problems

① 

$$\begin{cases} z(0) = z_0 \\ \frac{dz}{dt}(0) = 0 \end{cases}$$

② 

θ small

$$\begin{cases} \theta(0) = \theta_0 \\ \frac{d\theta}{dt}(0) = 0 \end{cases}$$

In general, these oscillators will be subject to some <u>damping</u> (air friction, energy loss by heating of the spring, etc..)

Their governing equations (c.f Newton's Law) are, respectively:

$$① \quad m\frac{d^2z}{dt^2} = \underbrace{-kz}_{\text{tension}} \underbrace{- \lambda\frac{dz}{dt}}_{\text{spring damping force}}$$

$$② \quad Lm\frac{d^2\theta}{dt^2} = \underbrace{-mg\theta}_{\text{assuming } \sin\theta \simeq \theta} \underbrace{-\lambda L\frac{d\theta}{dt}}_{\text{air friction force}}$$

In both cases, a natural unit for z & θ is given by their initial displacement, z_0 & θ_0. In the case of the pendulum, the natural unit length is L, while in the case of the oscillator the natural unit length is z_0 (since z has dimension of a length.).

However while in the case of the un-damped pendulum there was a single possibility to construct a unit of time, we now see that in both cases there are 2 possibilities:

Pendulum: as before we find that $\sqrt{\frac{L}{g}}$ is a natural unit for time, but now $\frac{m}{\lambda}$ is also one

Oscillator: $\sqrt{\frac{m}{k}}$ and $\frac{m}{\lambda}$ are both natural units of time

→ which one to choose?

Suppose the damping λ is small → then the oscillation timescale of the pendulum/ oscillator seems the most appropriate time unit to begin with. In choosing it, it can then be shown (cf homework) that both mechanical systems reduce to the same non-dimensional equation/ICs:

$$\begin{cases} \dfrac{d^2f}{dt^2} + f = -\varepsilon \dfrac{df}{dt} \\ f(0) = 1 \\ \dfrac{df}{dt}(0) = 0 . \end{cases}$$

Let's solve this problem using a perturbation expansion:

$$f(t) = f_0(t) + \varepsilon f_1(t) + \dots$$

→ 0th order $\begin{cases} \dfrac{d^2f_0}{dt^2} + f_0 = 0 \\ f_0(0) = 1 \\ \dfrac{df_0}{dt}(0) = 0 \end{cases}$ → $f_0(t) = \cos t$

1st order: $\begin{cases} \dfrac{d^2 f_1}{dt^2} + f_1 = \sin t \\ f_1(0) = 0 \\ \left.\dfrac{df_1}{dt}\right|_0 = 0 \end{cases}$

$\rightarrow \quad f_1(t) = A\cos t + B\sin t + Kt\cos t$

where $\quad K$ satisfies: $-2K\sin t - Kt\cancel{\cos t} + Kt\cancel{\cos t} = \sin t$ (from eq)

$\rightarrow \quad K = -\frac{1}{2}$

A satisfies: $\quad A = 0 \quad$ (from IC $f_1(0)=0$)

B satisfies: $\quad B + K = 0 \quad$ (from IC $\frac{df_1}{dt} = 0$)

$\rightarrow \quad B = \frac{1}{2}$

So $\quad f_1(t) = \frac{1}{2}\sin t - \frac{1}{2}t\cos t$

and so $\quad f(t) = \cos t + \frac{\varepsilon}{2}(\sin t - t\cos t)$

This is clearly a non-uniform expansion. Can we make it uniform using renormalization? Let's try with

$$t = \tau + \varepsilon\omega_1(\tau) + \varepsilon^2\omega_2(\tau) + \cdots$$

$$f(\tau) = \cos(\tau + \varepsilon\omega_1(\tau) + \ldots) + \frac{\varepsilon}{2}(\sin\tau - \tau\cos\tau) + \text{h.o.t}$$

$$= \cos\tau - \varepsilon\omega_1(\tau)\sin\tau + \frac{\varepsilon}{2}(\sin\tau - \tau\cos\tau) + \text{h.o.t}$$

The only way to make the term in τ disappear is to let $\omega_1(\tau) = -\frac{\tau}{2}\frac{\cos\tau}{\sin\tau}$. But this has <u>exactly</u> the same problem as what we found for the Van-der Pol oscillator! Clearly, since this is a linear problem, the issue cannot be coming from the presence of nonlinearities in the equations $\rightarrow$ it's something much more basic.

To understand what's going on, let's look for the <u>exact</u> solution to the problem.

The ODE has the characteristic eq: $r^2 + \varepsilon r + 1 = 0$

So $r = \dfrac{-\varepsilon \pm \sqrt{\varepsilon^2 - 4}}{2} = \dfrac{-\varepsilon \pm i\sqrt{4-\varepsilon^2}}{2}$ since ε is small

$= -\frac{\varepsilon}{2} \pm i\sqrt{1-\left(\frac{\varepsilon}{2}\right)^2}$

and therefore the general solution

$$f(t) = e^{-\frac{\varepsilon}{2}t}\left[A\cos\left(\sqrt{1-\tfrac{\varepsilon^2}{4}}\,t\right) + B\sin\left(\sqrt{1-\tfrac{\varepsilon^2}{4}}\,t\right)\right]$$

Fitting the BCs, we have: $A = 1$ and since

$$f'(t) = -\frac{\varepsilon}{2}e^{-\frac{\varepsilon}{2}t}\left[A\cos(\dots) + B\sin(\dots)\right]$$
$$+ e^{-\frac{\varepsilon}{2}t}\left[-\sqrt{1-\tfrac{\varepsilon^2}{4}}\,A\sin(\dots) + \sqrt{1-\tfrac{\varepsilon^2}{4}}\,B\cos(\dots)\right]$$

$f'(0) = 0 \Rightarrow -\frac{\varepsilon}{2}A + \sqrt{1-\frac{\varepsilon^2}{4}}\,B = 0$

So $B = \dfrac{\varepsilon}{2\sqrt{1-\frac{\varepsilon^2}{4}}} = \dfrac{\varepsilon}{\sqrt{4-\varepsilon^2}}$

and finally $f(t) = e^{-\frac{\varepsilon}{2}t}\left[\cos\left(\sqrt{1-\tfrac{\varepsilon^2}{4}}\,t\right) + \dfrac{\varepsilon}{\sqrt{4-\varepsilon^2}}\sin\left(\sqrt{1-\tfrac{\varepsilon^2}{4}}\,t\right)\right]$

We see that, with a correct strained coordinate (here $\tau = \sqrt{1-\frac{\varepsilon^2}{4}}\,t$) we can make the term in the square brackets into a uniform expansion in ε.

→ this is <u>not</u> the problem. However, there is clearly little we can do with the term in $e^{-\frac{\varepsilon}{2}t}$ →

it's own expansion will be $e^{-\frac{\varepsilon}{2}t} \simeq 1 - \frac{\varepsilon}{2}t + \frac{\varepsilon^2}{8}t^2 + \dots$

→ <u>this</u> is the source of non-uniformity in the problem! However, if we recognize that this problem is inherently of this form (a "rapid oscillation" times a "slowly decaying" amplitude), perhaps we should simply avoid expanding the first term, & leave it "as is".

→ The question is then, can we treat different physical processes differently in terms of their perturbation expansions, say, by expanding one & not the other? The answer is yes, provided the timescales intrinsic to the different processes are sufficiently different → i.e one is "little o" of the other.

(B) Multi-scale expansion (an introductory detour)

Consider a function that is quite similar to $f(t)$:

$g(t) = e^{-\varepsilon t} \sin t$

This function clearly has 2 typical timescales that are separated by a factor ε.

It's expansion in ε is non-uniform:

$$g(t) = \left(1 - \varepsilon t + \frac{\varepsilon^2 t^2}{2} + \dots\right) \sin t$$

However, if we recognize the two timescales, and let $T_0 = t$ and $T_1 = \varepsilon t$, g becomes a function of $\underline{\underline{2}}$ variables, T_0 & T_1:

$$g(T_0, T_1) = e^{-T_1} \sin(T_0)$$

→ this doesn't explicitly depend on ε, and does not need any expansion!

Suppose we now want to calculate the time-derivative of g:

$$\frac{dg}{dt} = -\varepsilon e^{-\varepsilon t} \sin t + e^{-\varepsilon t} \cos t .$$

Alternatively, recognizing that both T_0 & T_1 depend on t, we have

$$\frac{dg}{dt} = \frac{\partial g}{\partial T_0}\frac{dT_0}{dt} + \frac{\partial g}{\partial T_1}\frac{dT_1}{dt} = 1 \cdot e^{-T_1} \cos T_0 + \varepsilon \cdot \left(-e^{-T_1} \sin T_0\right)$$

$$= e^{-\varepsilon t} \cos t - \varepsilon e^{-\varepsilon t} \sin t \quad ✓$$

(c) Application of the multi-scale expansion to the damped linear oscillator problem

Going back to
$$\begin{cases} \dfrac{d^2f}{dt^2} + f = -\varepsilon\dfrac{df}{dt} \\ f(0) = 1 \\ \dfrac{df}{dt}(0) = 0 \end{cases}$$

let's now assume f is a function of 2 variables (T_0 and T_1) with $T_0 = t$, $T_1 = \varepsilon t$ and try to solve the problem from here.

As shown earlier, we have

$$\frac{df}{dt} = \frac{\partial f}{\partial T_0} + \varepsilon\frac{\partial f}{\partial T_1} \quad \text{so}$$

$$\frac{d^2f}{dt^2} = \left(\frac{\partial}{\partial T_0} + \varepsilon\frac{\partial}{\partial T_1}\right)\left(\frac{\partial f_0}{\partial T_0} + \varepsilon\frac{\partial f}{\partial T_1}\right) = \frac{d^2f}{dT_0^2} + 2\varepsilon\frac{d^2f}{dT_0 dT_1} + \varepsilon^2\frac{d^2f}{dT_1^2}$$

Plugging this back into the original system, we have

$$\begin{cases} \dfrac{\partial^2 f}{\partial T_0^2} + 2\varepsilon\dfrac{d^2f}{\partial T_0\partial T_1} + \varepsilon^2\dfrac{\partial^2 f}{\partial T_1^2} + f = -\varepsilon\left(\dfrac{\partial f}{\partial T_0} + \varepsilon\dfrac{\partial f}{\partial T_1}\right) \\ f(0,0) = 1 \\ \left(\dfrac{\partial f}{\partial T_0} + \varepsilon\dfrac{\partial f}{\partial T_1}\right)(0,0) = 0 . \end{cases}$$

If we then <u>also</u> assume that $f(T_0,T_1)$ is of the form $f(T_0,T_1) = f_0(T_0,T_1) + \varepsilon f_1(T_0,T_1) + \varepsilon^2 f_2(T_0,T_1) \ldots$

then, to zeroth order, we get

$$\begin{cases} \dfrac{\partial^2 f_0}{\partial T_0^2} + f_0 = 0 \\ f_0(0,0) = 1 \\ \dfrac{\partial f_0}{\partial T_0} = 0 . \end{cases}$$

$\leftarrow$ Be careful, this is now a Partial Differential Eq!

$\downarrow$

solution is of the form

$f_0(T_0,T_1) = A_0(T_1)\cos T_0 + B_0(T_1)\sin T_0$.

To fit the ICs, we then have $A_0(0) = 1$, and $B_0(0) = 0$ but this doesn't yet tell us what the whole functions $A_0(T_1)$ and $B_0(T_1)$ are.

This is found from the next order in expansion

To first order, we have:

$$\frac{\partial^2 f_1}{\partial T_0^2} + 2\frac{\partial^2 f_0}{\partial T_0 \partial T_1} + f_1 = -\frac{\partial f_0}{\partial T_0}$$

which can be rewritten as:

$$\frac{\partial^2 f_1}{\partial T_0^2} + f_1 = -\left[-A_0(T_1)\sin T_0 + B_0(T_1)\cos T_0\right] - 2\left[-\frac{\partial A_0}{\partial T_1}\sin T_0 + \frac{\partial B_0}{\partial T_1}\cos T_0\right].$$

f_1 will therefore be of the form

$$f_1(T_1, T_0) = A_1(T_1)\cos T_0 + B_1(T_1)\sin T_0 + \text{part. sol}.$$

But: note how unless the entire rhs is 0, the particular solution will contain terms of the kind $T_0 \sin T_0$ and $T_0 \cos T_0$ $\rightarrow$ $t\sin t$ and $t\cos t$

These are secular terms we would like to avoid.

Setting all the sources of secular terms to 0 in the rhs yields the desired equations for $A_0(T_1)$ and $B_0(T_1)$

$$\boxed{\text{Compatibility condition}} \rightarrow \begin{cases} 2\dfrac{\partial A_0}{\partial T_1} + A_0 = 0 & \rightarrow A_0(T_1) = A_0(0)e^{-\frac{T_1}{2}} = e^{-\frac{T_1}{2}} \\ 2\dfrac{\partial B_0}{\partial T} + B_0 = 0 & \rightarrow B_0(T_1) = B_0(0)e^{-\frac{T_1}{2}} = 0. \end{cases}$$

Putting this all together, we then find that

$$f(T_0, T_1) = e^{-\frac{T_1}{2}}\cos T_0 + O(\varepsilon)$$

$$\rightarrow \boxed{f(t) = e^{-\frac{\varepsilon t}{2}}\cos t + O(\varepsilon)}.$$

This solution also happens to be the correct multiscale expansion of the exact solution $f(t)$ with $T_0 = t$ and $T_1 = \varepsilon t$, to zeroth-order :

To see this :

$$f(t) = e^{-\frac{\varepsilon}{2}t}\left[\cos\left(\sqrt{1-\frac{\varepsilon^2}{4}}\,t\right) + \frac{\varepsilon}{\sqrt{4-\varepsilon^2}}\sin\left(\sqrt{1-\frac{\varepsilon^2}{4}}\,t\right)\right]$$

$$f(T_0,T_1) = e^{-\frac{T_1}{2}}\left[\cos\left(\sqrt{1-\frac{\varepsilon^2}{4}}\,T_0\right) + \underbrace{\frac{\varepsilon}{\sqrt{4-\varepsilon^2}}\sin\left(\sqrt{1-\frac{\varepsilon^2}{4}}\,T_0\right)}_{\text{h.o.t, neglect}}\right]$$

$$= e^{-\frac{T_1}{2}}\cos T_0 + O(\varepsilon) \quad \checkmark$$

Finally, suppose we wanted a 2-term multiscale expansion of f:

- We already have $f_1(T_0,T_1) = A_1(T_1)\cos T_0 + B_1(T_1)\sin T_0$ ... but
- This requires going to the next order to get compatibility conditions for $A_1(T_1)$ and $B_1(T_1)$ from the requirement that neither of them contain secular terms $\rightarrow$ homework.

<u>Notes</u> :

- As with all perturbation expansions, things work out if you have the right ansatz, but do not if you don't. Here it gets tricky because we first need to guess
  1. the right timescales to use
  2. the correct asymptotic sequence for f.

In many problems the correct choice for both ① and ② is far from obvious!

- Be careful to apply the ICs correctly (it's tricky with 2 timescales & a perturbation expansion)