II Lighthill's technique & the idea of renormalization

Lighthill Later came up with a generalization of the Linsted - Poincaré technique that can be used on much more general classes of problems. Let's first see it in action applied to the Duffing equation.

① The Duffing equation revisited

Lighthill's proposal is to create a strained coordinate τ that is more general than the one for the L-P technique, & relate it to t via

$$t = \tau + \varepsilon\omega_1(\tau) + \varepsilon^2\omega_2(\tau) + \dots$$

where $\omega_1(\tau)$, $\omega_2(\tau)$ ... are to be determined.

The Lindsted - Poincaré technique is just a special case of this expansion (cf. homework)

In that case, $\dfrac{d}{dt} = \dfrac{d}{d\tau}\cdot\dfrac{d\tau}{dt} = \dfrac{1}{\frac{dt}{d\tau}}\dfrac{d}{d\tau} = \dfrac{1}{1+\varepsilon\omega_1' + \varepsilon^2\omega_2' + \dots}\dfrac{d}{d\tau}$

Applying this to the Duffing eq, we get

$$\frac{1}{1+\varepsilon\omega_1'+\varepsilon^2\omega_2'+\dots}\frac{d}{d\tau}\left[\frac{1}{1+\varepsilon\omega_1'+\varepsilon^2\omega_2'+\dots}\frac{df}{d\tau}\right] - f = \varepsilon f^3$$

We then let $f = f_0(\tau) + \varepsilon f_1(\tau) + \dots$ as usual, so

$$\frac{1}{1+\varepsilon\omega_1'+\varepsilon^2\omega_2'+\dots}\frac{d}{d\tau}\left[\frac{1}{1+\varepsilon\omega_1'+\varepsilon^2\omega_2'+\dots}(f_0'+\varepsilon f_1'+\dots)\right]$$

$$- (f_0 + \varepsilon f_1 + \dots) = \varepsilon(f_0 + \varepsilon f_1 + \dots)^3$$

We then expand this in successive orders of ε and see what happens. Note that

$$\frac{1}{1+\varepsilon\omega_1'+\varepsilon^2\omega_2'+\dots} = 1 - (\varepsilon\omega_1' + \varepsilon^2\omega_2' + \dots) + (\varepsilon\omega_1' + \varepsilon^2\omega_2' + \dots)^2 + \dots$$

$$= 1 - \varepsilon(\omega_1' + \varepsilon\omega_2' + \dots) + \varepsilon^2(\omega_1' + \varepsilon\omega_2' + \dots)^2$$

$$= 1 - \varepsilon\omega_1' + \varepsilon^2(\omega_1'^2 - \omega_2') + \dots$$

so $(1 - \varepsilon\omega_1' + \varepsilon^2(\omega_1'^2 - \omega_2') + \dots)\frac{d}{d\tau}\left[(1 - \varepsilon\omega_1' + \varepsilon^2(\omega_1'^2 - \omega_2') + \dots)(f_0' + \varepsilon f_1' + \right.$

$- (f_0 + \varepsilon f_1 + \dots) = \varepsilon(f_0 + \varepsilon f_1 + \dots)^3$

To 0th order we recover $f_0'' - f_0 = 0$

If we require that $\quad = 0$ at $\quad = 0$ then we also need $\omega_1(\tau) = \omega_2(\tau) = \dots = 0$ at $\tau = 0$

In that case, the original initial conditions $f(0) = 1$, $\frac{df}{dt}(0) = 0$ become $f_0(0) = 1$, $\frac{df_0}{d\tau}(0) = 0$

$\longrightarrow \quad f_0(\tau) = \cos\tau$

To first order, we get

$$f_1'' - f_1 - \omega_1' f_0'' - (\omega_1' f_0')' = f_0^3$$

$$\longrightarrow \quad f_1'' - f_1 = -2\omega_1'\cos\tau - \omega_1''\sin\tau + \tfrac{1}{4}\cos 3\tau + \tfrac{3}{4}\cos\tau$$

Suppose we want, as in the Lindsted–Poincaré method, to eliminate the secular terms. They now arise from all the terms on the RHS in either $\sin\tau$ or $\cos\tau$ so we have to eliminate all of them: this requires

$$\omega_1''\sin\tau + 2\omega_1'\cos\tau - \tfrac{3}{4}\cos\tau = 0$$

with $\omega_1(0) = 0$.

$\rightarrow$ This is an equation for $\omega_1(\tau)$. Note that there are many solutions (2 general ones, + some arbitrariness since we only have one boundary condition); we only need to find one that satisfies our requirements, no need to find all of them. It's then fairly easy to see that if

$$\omega_1' = \frac{3}{8} \qquad (\omega_1'' = 0) \quad \text{then we're home.}$$

This implies $\omega_1(\tau) = \frac{3}{8}\tau$. The equation for f_1 then simplifies to

$$f_1'' - f_1 = \frac{1}{4}\cos 3\tau$$

We could proceed similarly for the next order.

<u>Note how</u>:

- ω_1' is actually the same as $-\omega_1$ we found in the Lindstedt-Poincaré Technique (this is not a coincidence, see homework)
- This method is <u>messy</u> & dealing with the $\frac{1}{1+\varepsilon\omega_1'+\cdots}$ fraction is tricky.
- To recover t from τ requires yet another expansion (and similarly to get $f(t)$)

Here: $f(\tau) = \cos\tau + \varepsilon f_1(\tau)$

$$t = \tau + \frac{3}{8}\varepsilon\tau + O(\varepsilon^2)$$

so
$$\begin{aligned}
\tau &= t - \frac{3}{8}\varepsilon\tau - O(\varepsilon^2) \\
&= t - \frac{3}{8}\varepsilon\left(t - \frac{3}{8}\varepsilon\tau + \cdots\right) - O(\varepsilon^2) \\
&\simeq t - \frac{3}{8}\varepsilon t + O(\varepsilon^2) \\
&= t\left(1 - \frac{3}{8}\varepsilon + O(\varepsilon^2)\cdots\right)
\end{aligned}$$

so $f(t) = \cos\left(t\left(1 - \frac{3}{8}\varepsilon + \cdots\right)\right) + \varepsilon f_1(\tau) \cdots$

② Renormalization

Because Lighthill's technique is so messy to use, a new method that is mathematically equivalent, but algebraically much more straightforward, was later proposed. The idea is to wait until we have non-uniform expansion in t, to introduce the new variable τ.

Recall from the last chapter that the non-uniform expansion for the Duffing equation, in the original variable t, is

$$f(t) = \cos t - \frac{\varepsilon}{8}\left[\frac{\cos 3t}{4} - \frac{\cos t}{4} - 3t\sin t\right] + O(\varepsilon^2).$$

Then let $t = \tau + \varepsilon\omega_1(\tau) + \varepsilon^2\omega_2(\tau) + \cdots$

$$\Rightarrow \quad f(\tau) = \cos\left[\tau + \varepsilon\omega_1(\tau) + \cdots\right] - \frac{\varepsilon}{8}\Big\{\frac{1}{4}\cos\left(3\tau + 3\varepsilon\omega_1(\tau) + \cdots\right)$$
$$- \frac{1}{4}\cos\left(\tau + \varepsilon\omega_1(\tau) + \cdots\right)$$
$$- 3\left(\tau + \varepsilon\omega_1(\tau) + \cdots\right)\cdot \sin\left(\tau + \varepsilon\omega_1(\tau) + \cdots\right)$$

We then expand this in ε at constant τ:

$$f(\tau) = \cos\tau - \varepsilon\omega_1(\tau)\sin\tau + \cdots$$
$$- \frac{\varepsilon}{8}\Big\{\frac{1}{4}\cos 3\tau + O(\varepsilon)$$
$$- \frac{1}{4}\cos\tau + O(\varepsilon)$$
$$- 3\tau\sin\tau + O(\varepsilon)\Big\} + O(\varepsilon^2)$$

To eliminate the secular term, we see that we simply have to set $\omega_1(\tau) = \frac{3}{8}\tau$

→ same result, a fraction of the time!

Of course we still have to invert t to get τ, but this is much simpler than before.

<u>Notes</u>: as for the Lindsted - Poincaré technique, the Lighthill method (with or without using renormalization) always has to go one step further in the expansion of the strained coordinate than in the expansion of the function to be consistent: <u>a 2-term expansion for τ is consistent with a 1-term expansion for $f(\tau)$</u>

When using renormalization, however, we <u>do</u> use the non-unif 2-term expansion in f in the <u>original</u> variable t to get the 2-term expansion for the strained coordinate. But once we get the latter, it is only consistent if used in conjuction with the <u>1-term</u> expansion of f in the new coordinate.

(3) Another worked example for a nonlinear oscillator

Consider
$$\frac{d^2 f}{dt^2} + f = \epsilon f\left(1 - \left(\frac{df}{dt}\right)^2\right)$$
with $f(0) = h$, $\frac{df}{dt}(0) = 0$.

To obtain a uniformly valid 1-term expansion for f we

- get a <u>non-uniform 2-term</u> expansion in t
- get a <u>2-term</u> expansion for τ
- use in the <u>one-term</u> expansion of $f(\tau)$.

let $f = f_0(t) + \varepsilon f_1(t) + \cdots$

$$\Rightarrow (f_0'' + \varepsilon f_1'' + \cdots) + (f_0 + \varepsilon f_1 + \cdots) = \varepsilon (f_0 + \varepsilon f_1 + \cdots)\left(1 - (f_0' + \varepsilon f_1' + \cdots)^2\right)$$

<u>0th order</u> $\Rightarrow$ $f_0'' + f_0 = 0$ with $f_0(0) = h$, $f_0'(0) = 0$

$$\Rightarrow f_0 = h\cos t$$

<u>1st-order</u>: $f_1'' + f_1 = f_0(1 - f_0'^2)$

$$= h\cos t\,(1 - h^2\sin^2 t)$$

$$= h\cos t - h^3(1-\cos^2 t)\cos t$$

$$= (h - h^3)\cos t + h^3\cos^3 t$$

$$= (h-h^3)\cos t + h^3\left(\tfrac{1}{4}\cos 3t + \tfrac{3}{4}\cos t\right)$$

$$= \frac{h^3}{4}\cos 3t + (h - \tfrac{1}{4}h^3)\cos t$$

So $f_1(t) = A\cos t + B\sin t + K_1\cos 3t + K_2 t\sin t$.

where K_1 satisfies: $-9K_1 + K_1 = \dfrac{h^3}{4} \Rightarrow K_1 = -\dfrac{h^3}{32}$

K_2 satisfies: $2K_2 = h - \frac{1}{4}h^3$

$$\Rightarrow K_2 = \frac{h}{2} - \frac{1}{8}h^3$$

Finally, to satisfy the BCs with $f_1(0) = 0$, $f_1'(0) = 0$

$$\Rightarrow \begin{cases} A + K_1 = 0 & \Rightarrow A = -K_1 = \dfrac{h^3}{32} \\ B = 0 \end{cases}$$

So $f_1(t) = \dfrac{h^3}{32}(\cos t - \cos 3t) + \dfrac{h}{2}\left(1 - \dfrac{h^2}{4}\right)t\sin t$

↑ secular term

and finally

$$f(t) = h\cos t + \varepsilon\left[\frac{h^3}{32}(\cos t - \cos 3t) + \frac{h}{2}\left(1 - \frac{h^2}{4}\right)t\sin t\right] + \cdots$$

We now let $t = \tau + \varepsilon\omega_1(\tau) + \cdots$ so that

$$f(\tau) = h\cos(\tau + \varepsilon\omega_1(\tau) + \cdots) + \varepsilon\Big[\frac{h^3}{32}\big(\cos(\tau + \varepsilon\omega_1(\tau) + \cdots) - \cos(3\tau + 3\varepsilon\omega_1(\tau) + \cdots)\big) + \frac{h}{2}\Big(1 - \frac{h^2}{4}\Big)(\tau + \varepsilon\omega_1(\tau) + \cdots) \cdot \sin(\tau + \varepsilon\omega_1(\tau) + \cdots$$

and expand:

$$f(\tau) = h\cos\tau - \varepsilon h\omega_1(\tau)\sin\tau + \varepsilon\Big[\frac{h^3}{32}(\cos\tau - \cos 3\tau) + \frac{h}{2}\Big(1 - \frac{h^2}{4}\Big)\tau\sin\tau\Big] + o(\varepsilon^2)$$

To eliminate the secular term, we simply let

$$h\omega_1(\tau)\sin\tau = \frac{h}{2}\Big(1 - \frac{h^2}{4}\Big)\tau\sin\tau \Rightarrow$$

$$\omega_1(\tau) = \frac{1}{2}\Big(1 - \frac{h^2}{4}\Big)\tau$$

so $t = \tau + \frac{\varepsilon}{2}\Big(1 - \frac{h^2}{4}\Big)\tau + \cdots$

so $\tau = t - \frac{\varepsilon}{2}\Big(1 - \frac{h^2}{4}\Big)t + \cdots$

Finally, $f(t) = h\cos\Big[\frac{2\pi t}{T}\Big]$ where

$$T = \frac{2\pi}{1 - \frac{\varepsilon}{2}\Big(1 - \frac{h^2}{4}\Big) + \cdots}$$ is the correct 1-term <u>uniform</u> expansion for f.

Oof! To get the next-order correction, we'd have to get the 3rd-term in $f(t)$ (non-uniform), then get ω_2, then plug that into $f_1(\tau)$ ...

Note, however, that for periodic systems, the Lindsted-Poincaré technique works just as well, and is probably a little more straightforward still. It is up to you to decide which one you prefer to use.