CHAPTER 3

In this Chapter we begin our tour of methods that will be used to address some of the problems discovered in the previous Chapters when trying to solve equations using asymptotic expansions.

Here we focus on the issue of non-uniform expansions, and learn 2 methods: the Lindstedt-Poincaré method which applies to periodic systems that are slightly modified by a small perturbations, and the Lighthill technique that generalizes the Lindsted-Poincaré method for a wider class of problems.

I The Linsted-Poincaré method for periodic systems

① General Idea

Consider the two following periodic functions

$$f(t) = \cos(t+\varepsilon) \quad \text{and} \quad g(t) = \cos((1+\varepsilon)t+\varepsilon)$$

Let's expand them in powers of ε:

$$f(t) = \cos t - \varepsilon \sin t - \frac{\varepsilon^2}{2}\cos t + \frac{\varepsilon^3}{6}\sin t + \dots$$

$$g(t) = \cos(t+\varepsilon(1+t))$$

$$= \cos t - \varepsilon(1+t)\sin t - \frac{\varepsilon^2(1+t)^2}{2}\cos t + \dots$$

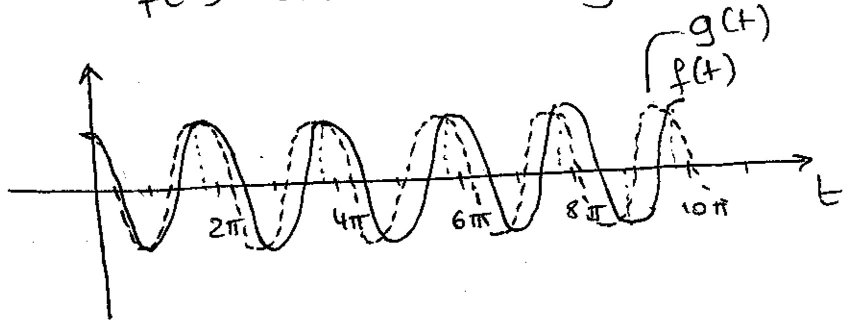
The expansion for $f(t)$ is clearly uniform, while the one for $g(t)$ is clearly non-uniform and contains secular terms of progressively higher powers in t

Note, however, that if we had defined a new variable $\tau = (1+\varepsilon)t$, then a new function $G(\tau;\varepsilon)$ as $G(\tau;\varepsilon) = g(t) = \cos(\tau+\varepsilon)$ then the expansion for G becomes uniform: $G(\tau;\varepsilon) = \cos\tau - \varepsilon\sin\tau - \frac{\varepsilon^2}{2}\cos\tau \ldots$

→ A change of variable seems to fix the problem!

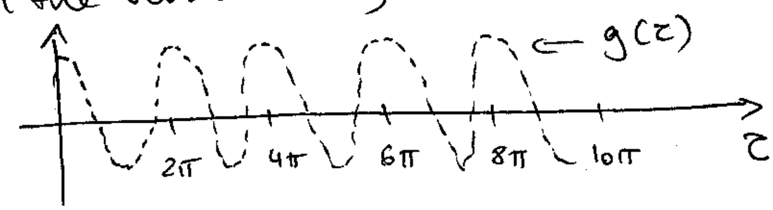
This seems somewhat too good to be true but in this case is indeed true, & lies at the heart of the Linsted-Poincaré technique. The reason why it works lies in the reason for the original non-uniformity:

⇒ $f(t)$ & $g(t)$ have slightly different periods (that of $f(t)$ is 2π while that of $g(t)$ is $\frac{2\pi}{1+\varepsilon}$) but their expansions are both, to lowest order, in $\cos t$ (with period 2π). This is a good approximation for $f(t)$ but eventually becomes bad for $g(t)$



after $O(\frac{1}{\varepsilon})$, $f(t)$ & $g(t)$ are different by $O(1)$ due to the secular terms / mismatch in period.

On the other hand if the x-axis is rescaled ever so slightly by changing $t \to \tau = (1+\varepsilon)t$ then g becomes periodic with period 2π as well. (in the variable τ)



Another way of interpreting the issue is that in writing $g(t) \simeq \cos t - \varepsilon(1+t)\sin t - \frac{\varepsilon^2}{2}(1+t)^2\cos t + \cdots$ we assume $\varepsilon \to 0$ at fixed t $\to$ but for large enough t, $\varepsilon(1+t)$ will always become $O(1)$ & the series no longer converges.

On the other hand $g(\tau) \simeq \cos \tau - \varepsilon \sin \tau - \frac{\varepsilon^2}{2}\cos\tau + \cdots$ at fixed τ, $\varepsilon \to 0$, does converge for all τ. What happens is that t and τ eventually become quite different from one another. $t - \tau = t - (1+\varepsilon)t = -\varepsilon t$

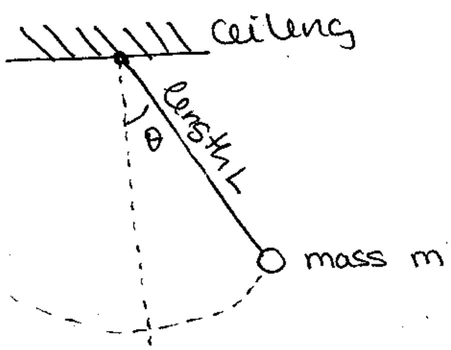
For $t = O(\frac{1}{\varepsilon})$, t & τ become $O(1)$ different.

2) Example of application: the Duffing oscillator

We first learned of the Duffing oscillator in the last lecture. We will now see where it naturally arises, & how to resolve the secular term problem encountered when trying to find asymptotic solutions of the equation.

a Model

Consider a normal pendulum with no friction:

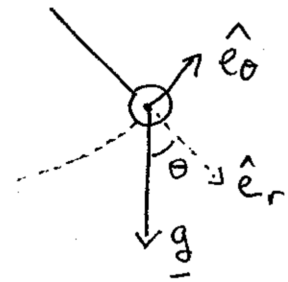


This pendulum is subject to 2 forces: gravity $\underline{g} = -g\,\hat{e}_z$ and tension from the string $\underline{T}$ always in the direction of the string.

Using Newton's Law $\underline{F} = m\underline{a}$, and projecting it in the direction perpendicular to the string & tangential to the motion (i.e. on the circle of length L), we have

$$\cancel{m}L\frac{d^2\theta}{dt^2} = -\cancel{m}g\sin\theta$$

(Note how m disappears from problem)



Zoom near mass m.

Let's also assume that the pendulum is released from rest and from angle θ_0 at $t=0 \Rightarrow$ $\theta(0) = \theta_0$

$$\frac{d\theta}{dt}(0) = 0$$

b. <u>Non-dimensionalization and expansion for small θ_0</u>

- A natural unit for the angle is θ_0 so let

$$f = \frac{\theta}{\theta_0} \quad \rightarrow \quad \theta = \theta_0 f$$

- L is a natural unit for length, and that leaves g, which has units of length/time^2

$\rightarrow$ a natural unit of time is $\sqrt{\frac{L}{g}}$ so let $\hat{t} = t\sqrt{\frac{g}{L}}$

Then $L\frac{d^2\theta}{dt^2} = -g\sin\theta$ becomes

$$\frac{L\theta_0 g}{L}\frac{d^2 f}{d\hat{t}^2} = -g\sin(\theta_0 f)$$

$$\Rightarrow \quad \frac{d^2 f}{d\hat{t}^2} = -\frac{\sin(\theta_0 f)}{\theta_0}$$

Finally, let's assume θ_0 is small so

$$\frac{d^2 f}{d\hat{t}^2} \simeq -\frac{(\theta_0 f - \theta_0^3 f^3/6 + \cdots)}{\theta_0} = -f + \frac{\theta_0^2 f^3}{6} - \cdots$$

$$\frac{d^2 f}{d\hat{t}^2} \simeq -f + \varepsilon f^3 \quad \text{if} \quad \varepsilon = \frac{\theta_0^2}{6}$$ (which is small since θ_0 is.)

This is the Duffing equation we saw last Chapter!
At that point, we found an expansion in ε of the kind $f(t) = \cos t - \frac{\varepsilon}{8}\left[\frac{\cos 3t}{4} - \frac{\cos t}{4} - 3t \sin t\right] + o(\varepsilon^2)$

↑ source of non uniformity

However, having f increasing secularly with time is clearly unphysical → a pendulum's amplitude doesn't spontaneously start increasing.

This example is clearly a candidate for the method of strained coordinates by Lindsted & Poincaré.

c. Method of strained coordinates

The idea is to assume the existence of a "better" coordinate τ, and let it have the form

$$\tau = t(1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots) \quad \text{with } \{\omega_i\} \text{ TBD}$$

let's see what this buys us

$$\frac{d}{dt} = \frac{d}{d\tau}\frac{d\tau}{dt} = (1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots)\frac{d}{d\tau}$$

So our original equation becomes

$$(1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots)^2 \frac{d^2 f}{d\tau^2} = -f + \varepsilon f^3$$

Next, let's assume as usual an expansion for f as $f(\tau;\varepsilon) = f_0(\tau) + \varepsilon f_1(\tau) + \varepsilon^2 f_2(\tau) + \cdots$

$$\rightarrow \quad (1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots)^2 (f_0'' + \varepsilon f_1'' + \cdots) = -(f_0 + \varepsilon f_1 + \cdots) + \varepsilon(f_0 + \varepsilon f_1 + \cdots)^3$$

To 0th order we get

$$f_0'' = -f_0$$

At $t=0$, $\tau = 0$ and so $f(0)=1$, $f'(0)=0$

implies $f_0(0)=1$, $f_0'(0)=0$

$$\Rightarrow \quad f_0(\tau) = \cos\tau .$$

To 1st order, we get

$$2\omega_1 f_0'' + f_1'' = -f_1 + f_0^3$$

so

$$f_1'' + f_1 = -2\omega_1 f_0'' + f_0^3$$

$$= +2\omega_1 \cos\tau + (\cos\tau)^3$$

$$= 2\omega_1 \cos\tau + \frac{1}{4}\cos 3\tau + \frac{3}{4}\cos\tau$$

$$= \frac{1}{4}\cos 3\tau + \left(2\omega_1 + \frac{3}{4}\right)\cos\tau$$

So far ω_1 was unspecified so we are free to choose it as we wish. Here we see that if we take $\omega_1 = -\frac{3}{8}$, then this "kills" the term $\cos\tau$ on the RHS, which was itself at the origin of the secular terms in the originally-expanded solution $\Rightarrow$ if $\omega_1 = -\frac{3}{8}$ then

$$f_1'' + f_1 = \frac{1}{4}\cos 3\tau \quad \text{then}$$

$$f_1(\tau) = A\cos\tau + B\sin\tau + K\cos 3\tau$$

with K such that $-9K + K = \frac{1}{4} \Rightarrow K = -\frac{1}{32}$

ICs: $f_1(0) = 0 = \frac{df_1}{d\tau} = 0$ so

$$\begin{cases} A + K = 0 \\ B = 0 \end{cases} \Rightarrow \begin{cases} A = -K = \frac{1}{32} \\ B = 0 \end{cases}$$

So finally, $f_1(\tau) = \frac{1}{32}(\cos\tau - \cos 3\tau)$

and $f(\tau) = \cos\tau + \frac{\varepsilon}{32}(\cos\tau - \cos 3\tau) + \text{h.o.t}$

$\Rightarrow$ so far, $f(t) = \cos\left(\left(1-\frac{3}{8}\varepsilon\right)t\right) + \frac{\varepsilon}{32}\left(\cos\left(1-\frac{3}{8}\varepsilon\right)t - \cos 3\left(1-\frac{3}{8}\varepsilon\right)t\right)$

$+ \text{h.o.t}$

d. How far to go in the expansion? Consistency check

Let's inspect the neglected "higher order terms". These essentially contain $\varepsilon^2 f_2(t) + \varepsilon^3 f_3(t) + \cdots$

$f_2(t)$ satisfies

$$2\omega_2 f_0'' + \omega_1^2 f_0'' + 2\omega_1 f_1'' + f_2'' = -f_2 + 3f_0^2 f_1$$

which can be arranged into

$$f_2'' + f_2 = 3\cos^2\tau \cdot \frac{1}{32}(\cos\tau - \cos 3\tau)$$
$$+ (2\omega_2 + \omega_1^2)\cos\tau$$
$$+ \frac{2\omega_1}{32}(\cos\tau - 9\cos 3\tau)$$

As before only the terms in $\cos\tau$ on the RHS are problematic, but note that if they are present then $f_2(t)$ will also contain secular terms. In that case, we would have

$$f(\tau) = \cos\tau + \frac{\varepsilon}{32}(\cos\tau - \cos 3\tau) + \varepsilon^2\tau \cdot f_{2S}(\tau)$$
$$+ \varepsilon^2 f_{2NS}(\tau) + \text{h.o.t}$$

where f_{2S} are secular terms and f_{2NS} are non-secular terms.

$\rightarrow$ If we don't get rid of the f_2 secular then the expansion we wrote earlier is non-uniform!

[This shows that if we want a consistent expansion to $O(\varepsilon)$ in f, we need to expand to $O(\varepsilon^2)$ in the strained coordinate.

Noting that $\cos^2\tau \cos 3\tau = \frac{1}{2}(1+\cos 2\tau)\cos 3\tau$

$$= \frac{1}{2}\cos 3\tau + \frac{1}{4}(\cos 5\tau + \cos\tau)$$

and $\cos^3\tau = \frac{1}{4}\cos 3\tau + \frac{3}{4}\cos\tau$

then the secular terms at this order would come from the RHS term

$$\frac{3}{32}\left[\frac{3}{4}\cos\tau - \frac{1}{4}\cos\tau\right] + (2\omega_2 + \omega_1^2)\cos\tau + \frac{2\omega_1}{32}\cos\tau$$

To eliminate it, we require that

$$\frac{3}{64} + (2\omega_2 + \omega_1^2) + \frac{2\omega_1}{32} = 0$$

$$\Leftrightarrow \quad \omega_2 = \left[-\frac{3}{64} + \frac{2}{32}\cdot\frac{3}{8} - \left(\frac{3}{8}\right)^2\right]\cdot\frac{1}{2} = -\frac{21}{256}$$

The correct uniform expansion for $f(t)$ is therefore

$$\begin{cases} f(\tau) = \cos\tau + \frac{\varepsilon}{32}(\cos\tau - \cos 3\tau) + \varepsilon^2 f_2(\tau) + \cdots \\ \tau = t\left(1 - \frac{3}{8}\varepsilon - \frac{21}{256}\varepsilon^2 + \cdots\right) \end{cases}$$

Note that, by a similar argument, if we want to go to 3^{rd} order in $f(\tau)$ (i.e by actually calculating $f_2(\tau)$ we would have to find ω_3 to be consistent & avoid non-uniformities.

Based on this expansion, we then see that the new period of the pendulum is

$$T = \frac{2\pi}{1 - \frac{3}{8}\varepsilon - \frac{21}{256}\varepsilon^2 + \dots}$$

See homework for more practice problems.

Notes: 1. As usual, it isn't obvious at all that the correct asymptotic sequence in ε that one should use is $\{1, \varepsilon, \varepsilon^2, \varepsilon^3, \dots\}$ for f and for the strained coordinate. In fact, problems where other sequences are needed certainly exist. Be on the lookout for them if the obvious expansion doesn't work.

2. The Lindsted-Poincaré technique only works for periodic problems whose period is the only thing affected by the perturbations. Problems where the perturbation introduces time-dependent modulations of the amplitude of oscillations cannot be treated this way

3. The Lindsted - Poincaré method can only be used in the case of IVPs, but not for BVPs. This should be obvious from the fact that the second boundary needs to remain fixed in BVP problems, and NOT depend on ε.