## 4) Convergent vs divergent series

To determine whether any series of the kind $\sum_{n=0}^{\infty} a_n$ converges, it suffices to show that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$.

This also then implies that $\lim_{n\to\infty} |a_n| = 0$. Hence, for a Taylor series to converge, we simply need to show that the remainders satisfy $\lim_{n\to\infty} \left|\frac{R_{n+1}}{R_n}\right| < 1$

Non-convergent series are not as big a problem as they may seem, however. Indeed, in perturbation methods we usually just keep a few terms in the series instead of seeking the limit $N \to \infty$. For instance, if we choose to only use the first 3 terms, we would write

$$f(x) \simeq f(x_0) + (x-x_0) f'(x_0) + \frac{(x-x_0)^2}{2} f''(x_0) + R_2$$

In that case, $R_2$ serves as an estimate of the error made in the approximation. While the remainder may not <u>always</u> tend to 0 as more terms are kept, it may still be small enough for the purposes of our approximate work.

See examples in pages 26-33 of textbook for more on this topic.

# II.3 Little o symbol, and asymptotic sequences

① Little o

- Earlier we defined the $O$ "big o" symbol to relate two functions satisfying:

$$\text{if} \quad \lim_{x\to x_0} \frac{f(x)}{g(x)} = C \quad (C\neq 0) \quad \text{then} \quad f(x) = O(g(x)) \text{ as } x\to x_0 .$$

- If, on the other hand

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = 0 \quad \text{then} \quad f(x) = o(g(x)) \text{ as } x\to x_0$$

little o.

This loosely means that $f(x)$ tends to zero much faster than $g(x)$ as $x\to x_0$.

Examples:

- $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$ so $\sin x = O(x)$ as $x\to 0$

but $\lim\limits_{x\to 0} \dfrac{\sin x}{\sqrt{x}} = 0$ so $\sin x = o(\sqrt{x})$ as $x\to 0$.

- $\lim\limits_{x\to\infty} \dfrac{x}{e^x} = 0$ so $x = o(e^x)$ as $x\to +\infty$

and in fact, for <u>any</u> $N$, any $a>1$

$$\lim_{x\to\infty} \frac{x^N}{a^x} = 0 \quad \text{so} \quad x^N = o(a^x) \text{ as } x\to +\infty .$$

Similarly

- $\lim\limits_{x\to 0} x\ln x = 0$ as $x\to 0$ so

$$x = o\left(\frac{1}{\ln x}\right) \text{ as } x\to 0$$

and in fact, for any $N$, and $a>1$

$$\lim_{x\to 0} x^N \log_a x = 0 \quad \text{so} \quad x^N = o\left(\frac{1}{\log_a x}\right) \text{ as } x\to 0.$$

② Asymptotic sequences & series

Based on the definition of $o$, we can then define an asymptotic sequence as any combination of functions $\{\delta_0(\varepsilon), \delta_1(\varepsilon), \delta_2(\varepsilon), \delta_3(\varepsilon), \dots\}$ such that, $\forall n$, $\delta_{n+1}(\varepsilon) = o(\delta_n(\varepsilon))$

Example: • The sequence of functions

$$\{1, \varepsilon, \varepsilon^2, \varepsilon^3, \varepsilon^4, \dots\}$$

is an asymptotic sequence because $\varepsilon^{N+1} = o(\varepsilon^N)\ \forall N$.

• The sequence

$$\{1, \varepsilon^{1/2}, \varepsilon, \varepsilon^{3/2}, \dots\}$$

is an asymptotic sequence because $\varepsilon^{\frac{N+1}{2}} = o(\varepsilon^{\frac{N}{2}})\ \forall N$

Less obvious ones:

• The sequence

$$\{1, \ln(1+\varepsilon), \ln(1+\varepsilon^2), \ln(1+\varepsilon^3), \dots\}$$

is an asymptotic sequence.

To see this note that

$$\lim_{\varepsilon \to 0} \frac{\ln(1+\varepsilon^{N+1})}{\ln(1+\varepsilon^N)} = \lim_{\varepsilon \to 0} \frac{\varepsilon^{N+1}}{1+\varepsilon^{N+1}} \cdot \frac{1+\varepsilon^N}{\varepsilon^N}$$

(using l'Hopital's rule)

$$= 0 \quad \forall N.$$

Having defined an asymptotic sequence, note that a function $f(x)$ can, under some circumstances (see below) be written near $x_0$ as the asymptotic series

$$f(x) = \sum_{n=0}^{N-1} a_n \delta_n(x-x_0) + R_N$$

remainder

• Note that the selection of the sequence itself is not unique, that is, we could choose many different sequences $\{\delta_n\}$ in the formula above. However, if an appropriate sequence has been selected, then the coefficients $a_n$ are unique to that sequence.

For instance: If we want to approximate $\sin\varepsilon$ with the sequence $\{1, \varepsilon, \varepsilon^2, \varepsilon^3, \varepsilon^4, \varepsilon^5 \dots\}$ we get the asymptotic series

$$\sin\varepsilon = \varepsilon - \frac{\varepsilon^3}{3!} + \frac{\varepsilon^5}{5!} \quad \Rightarrow \quad \begin{aligned} a_0 &= 0 \\ a_1 &= 1 \\ a_2 &= 0 \\ a_3 &= -\frac{1}{3!} \\ a_4 &= 0 \\ a_5 &= \frac{1}{5!} \\ &\text{etc.} \end{aligned}$$

If, on the other hand, the selected sequence was

$$\{\varepsilon, \varepsilon^3, \varepsilon^5, \varepsilon^7 \dots\} \text{ then } \quad \begin{aligned} a_0 &= 1 \\ a_1 &= -\frac{1}{3!} \\ a_2 &= \frac{1}{5!} \\ &\text{etc...} \end{aligned}$$

This then shows that while the selection of an appropriate sequence for a function is not unique, not all sequences are appropriate either:

Example: $\{1, \varepsilon^2, \varepsilon^4, \varepsilon^6, \dots\}$ is NOT a good sequence to use to represent $\sin\varepsilon$.

Proof of uniqueness: Suppose $f(\varepsilon) = \sum_{n=0}^{N} a_n \delta_n(\varepsilon) + R_N$

then $$f(\varepsilon) = a_0 \delta_0(\varepsilon) + \sum_{n=1}^{N} a_n \delta_n(\varepsilon) + R_N$$

so $$\lim_{\varepsilon\to 0} \frac{f(\varepsilon)}{\delta_0(\varepsilon)} = a_0 + \sum_{n=1}^{N} a_n \lim_{\varepsilon\to 0} \frac{\delta_n(\varepsilon)}{\delta_0(\varepsilon)} + \lim_{\varepsilon\to 0} \frac{R_N}{\delta_0(\varepsilon)}$$

$$= a_0 + 0 + 0 \leftarrow \text{by definition}$$

Hence $a_0$ is uniquely determined to be

$$a_0 = \lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{\delta_0(\varepsilon)}.$$

Similarly, it's easy to show that

$$a_1 = \lim_{\varepsilon \to 0} \frac{f(\varepsilon) - a_0\delta_0(\varepsilon)}{\delta_1(\varepsilon)}$$

$$\vdots$$

$$a_n = \lim_{\varepsilon \to 0} \frac{f(\varepsilon) - \sum_{k=0}^{n-1} a_k \delta_k(\varepsilon)}{\delta_n(\varepsilon)}$$

## (3) Uniform & non-uniform expansions

Suppose now that the function $f(x)$ is solution of an equation containing the small parameter $\varepsilon$ (as in the first example of Chapter 1). Then the solution, ideally, should be expanded as

$$f(x;\varepsilon) = \sum_{n=0}^{N} a_n(x)\,\delta_n(\varepsilon) + R_N(x;\varepsilon)$$

where $R_N(x,\varepsilon) = O(\delta_{n+1}(\varepsilon))$

where $\{\delta_n(\varepsilon)\}$ is an asymptotic sequence for small $\varepsilon$.

Here the coefficients of the series representing $f$ are themselves functions of the independent variable.

### Examples

- In chapter 1, we had

$$w(t;\varepsilon) = -t + \varepsilon\frac{t^2}{2} - \varepsilon^2\frac{t^3}{6} + \cdots$$

- Suppose the exact solution of an equation is $f(x;\varepsilon) = \dfrac{1}{1-\varepsilon\sin x}$

Then $f(x,\varepsilon) = 1 + \varepsilon\sin x + \varepsilon^2\sin^2 x + \varepsilon^3\sin^3 x + \cdots$

The behaviour of these two series is very different, however.

In the first case, for large $t$, we see that the coefficients in front of $S_n(\varepsilon)$ increase with $n$. In other words, there are values of $t$ for which the series becomes divergent:

$$\left|\frac{R_{N+1}}{R_N}\right| \propto \frac{\varepsilon^{N+1}\, t^{N+2}}{\varepsilon^{N}\, t^{N+1}} \propto \varepsilon t \quad \text{(ignoring the constant coefficient)}$$

$\rightarrow$ if $t > \frac{K}{\varepsilon}$ then $\lim_{N\to\infty} \left|\frac{R_{N+1}}{R_N}\right|$ is NOT $<1$.

$\Longrightarrow$ In this case, the series is not a uniform asymptotic expansion in the sense that it is convergent for some values of $t$, but not others.

Definition: A uniform asymptotic expansion is one in which the remainder $R_N(x,\varepsilon)$ satisfies

$$|R_N(x;\varepsilon)| \leq K\, S_{N+1}(\varepsilon)$$

$\uparrow$ a constant independent of $x$.

In this example we have $R_N(x;\varepsilon) \sim K t^{N+1} \varepsilon^N$
$\rightarrow$ this cannot be bounded independently of $t$.

However, in the second example

$$f(x;\varepsilon) = 1 + \varepsilon \sin x + \varepsilon^2 \sin^2 x + \cdots$$

then $R_N \sim \varepsilon^3 \sin^3 x$

$$\rightarrow |R_N| \sim \varepsilon^3 |\sin^3 x| \leq \varepsilon^3$$

$\uparrow$ CAN be bounded

So the series is uniformly convergent, and is a uniform asymptotic expansion.

## (4) Typical sources of non-uniformity in asymptotic expansions

There are two typical sources of non-uniformity in an expansion

- infinite domains, which allow "large values" of the independent variable to affect the convergence of the series (as in Chapter 1)
- Singularities in the governing equations as $\varepsilon \to 0$ (eg. existence of singular expansions)

Example 1: Infinite domain

Let's consider a nonlinear oscillator of the form

$$\frac{d^2 f}{dt^2} = -f(1+\varepsilon f^2) \quad \leftarrow \text{the Duffing equation.}$$

This equation commonly arises in mechanical & electrical systems (See later for more detail).

Let's assume for the moment that $f(0) = h_0$ and $\left.\frac{df}{dt}\right|_{t=0} = 0$.

Suppose that solutions exist of the form

$$f = f_0(t) + \varepsilon f_1(t) + \varepsilon^2 f_2(t) + \cdots$$

then, order by order, we get

$$\frac{d^2 f_0}{dt^2} = -f_0 \quad \text{with} \quad \begin{cases} f_0(0) = h_0 \\ \frac{df_0}{dt}(0) = 0 \end{cases}$$

$$\frac{d^2 f_1}{dt^2} = -f_1 - f_0^3 \qquad \text{with} \quad \begin{cases} f_1(0) = 0 \\ \frac{df_1(0)}{dt} = 0 \end{cases}$$

$$\frac{d^2 f_2}{dt^2} = -f_2 - 3f_0^2 f_1 \qquad \text{etc} \ldots$$

The solution to the zeroth order with $f_0(0) = h_0$ is

$$f_0(t) = h_0 \cos t$$

Then we have

$$\frac{d^2 f_1}{dt^2} = -f_1 - h_0^3 (\cos t)^3 = -f_1 - h_0^3 \left( \frac{1}{4} \cos 3t + \frac{3}{4} \cos t \right)$$

The general solution to the homogeneous problem is

$$f_1^G(t) = A \cos t + B \sin t$$

For the particular solution, the term in $\cos 3t$ simply suggest another term in $\cos 3t$. However, the one in $\cos t$ requires more care, & suggests a solution of the kind $t \sin t$.

$\rightarrow$ we try $\quad f_1^{PS}(t) = K_1 \cos 3t + K_2 t \sin t$.

then

$$\frac{df_1^{PS}}{dt} = -3K_1 \sin 3t + K_2 \sin t + K_2 t \cos t$$

$$\frac{d^2 f_1^{PS}}{dt^2} = -9K_1 \cos 3t + 2K_2 \cos t - K_2 t \sin t$$

$$\Rightarrow \quad \frac{d^2 f_1^{PS}}{dt^2} = -f_1^{PS} - \frac{h_0^3}{4} \cos 3t - \frac{3h_0^3}{4} \cos t$$

$$\Leftrightarrow \quad -9K_1 \cos 3t + 2K_2 \cos t - \cancel{K_2 t \sin t}$$
$$= -(K_1 \cos 3t + \cancel{K_2 t \sin t}) - \frac{h_0^3}{4} \cos 3t - \frac{3h_0^3}{4} \cos t$$

$$\Rightarrow \quad -9K_1 = -K_1 - \frac{h_0^3}{4} \qquad 2K_2 = -\frac{3h_0^3}{4}$$

so $K_1 = \frac{h_0^3}{32}$ $\qquad K_2 = -\frac{3h_0^3}{8}$

and so $f_1(t) = A\cos t + B\sin t + \frac{h_0^3}{32}\cos 3t - \frac{3h_0^3}{8} t \sin t$

→ to satisfy the IC:

$$f_1(0) = 0 \qquad \frac{df_1}{dt} = 0 \qquad \begin{cases} A + \frac{h_0^3}{32} = 0 \\ B = 0 \end{cases}$$

and finally, $f_1(t) = \frac{h_0^3}{32}(\cos 3t - \cos t) - \frac{3h_0^3}{8} t \sin t$

so

$$f(t) = h_0 \cos t + \frac{\varepsilon h_0^3}{8}\left[\frac{\cos 3t}{4} - \frac{\cos t}{4} - 3t \sin t\right] + O(\varepsilon^2)$$

This expansion, however, is not uniform.
Suppose the first term is the one kept, and the second is the remainder, then we see that the latter <u>cannot</u> be bounded in a way that is independent of t because of the $t\sin t$ term.

This term is called a "<u>secular</u>" term. It becomes large as soon as t becomes of the order of $\frac{1}{\varepsilon}$
→ when that is the case, the remainder becomes of the same order as $h_0 \cos t$.

In the next chapter, we will learn how to deal with secular terms.

Example 2 Small parameter multiplying highest derivative → singular limits.

In the previous chapter, we saw how small parameters multiplying the highest-order term in a polynomial lead to the emergence of singular solutions. Let's see what it does for ODEs.

Consider the very simple ODE $\varepsilon \frac{df}{dx} + f = e^{-x}$ where $\varepsilon$ is small and positive, with $f(0) = 2$.

Assuming an expansion of the form

$$f = f_0(x) + \varepsilon f_1(x) + \varepsilon^2 f_2(x) \ldots$$

then, to lowest order and very naively, we would get

$$f_0(x) = e^{-x} \quad \text{with boundary condition } f_0(0) = 2$$

→ clearly that's impossible; similarly to the next order we would just get $\frac{df_0}{dx} + f_1 = 0$

$$\Rightarrow \quad f_1 = -\frac{df_0}{dx} = e^{-x}$$

However, the BC to apply here would be $f_1(0) = 0$

→ again clearly impossible.

Let's now try to be smarter & remember that we should expect a singular expansion based on our discussion of the last chapter. In that case, let's assume

$$f(x) = \frac{1}{\varepsilon} f_{-1} + f_0 + \varepsilon f_1 + \ldots$$

then

$$\begin{cases} \text{To order } \frac{1}{\varepsilon} \Rightarrow & f_{-1} = 0 \quad \rightarrow \text{already problematic} \\ \text{To order } 1 \Rightarrow & \frac{df_{-1}}{dx} + f_0 = e^{-x} \\ \quad \vdots \end{cases}$$

→ It looks like the solution found in the polynomial case does not work here. Why?

Since we can, actually, find the exact solution let's look at it to see what's going wrong.

Rewriting the equation as

$$\frac{df}{dx} = -\frac{1}{\varepsilon}f + \frac{1}{\varepsilon}e^{-x}$$ we see that

$$f(x) = Ke^{-\frac{x}{\varepsilon}} + f^{PS}$$, where

$f^{PS}$ is of the form $f^{PS}(x) = Ae^{-x}$.

Plugging it in, we get

$$-A\cancel{e^{-x}} = -\frac{1}{\varepsilon}A\cancel{e^{-x}} + \frac{1}{\varepsilon}\cancel{e^{-x}}$$

$$\Rightarrow \left(\frac{1}{\varepsilon}-1\right)A = \frac{1}{\varepsilon} \quad \Rightarrow \quad A = \frac{1}{1-\varepsilon}$$

so $$f(x) = Ke^{-x/\varepsilon} + \frac{1}{1-\varepsilon}e^{-x}$$

Finally, to fit the IC, we get $2 = K + \frac{1}{1-\varepsilon}$ so

$$K = 2 - \frac{1}{1-\varepsilon} = \frac{2-2\varepsilon-1}{1-\varepsilon} = \frac{1-2\varepsilon}{1-\varepsilon}$$

So $$f(x;\varepsilon) = \frac{1-2\varepsilon}{1-\varepsilon}e^{-\frac{x}{\varepsilon}} + \frac{1}{1-\varepsilon}e^{-x}$$

Expanding this in powers of $\varepsilon$, we then get

$$f(x;\varepsilon) = (1-2\varepsilon)(1+\varepsilon+\varepsilon^2+\dots)e^{-\frac{x}{\varepsilon}} + (1+\varepsilon+\varepsilon^2+\dots)e^{-x}$$

↑ but what to do with this?

$\Rightarrow$ $e^{-x/\varepsilon}$ doesn't have an expansion in powers of $\varepsilon$.

→ It's no surprise our attempt to write $f$ as an asymptotic sequence in $\varepsilon$ failed.

However, we can still learn something from this.

For $x \gg \varepsilon$, the term in $e^{-x/\varepsilon}$ is exponentially small hence

$$f(x;\varepsilon) \simeq (1+\varepsilon+\varepsilon^2+\dots)\,e^{-x}$$

→ so the asymptotic sequence should work then. The key is not to worry about the initial conditions (yet) since the latter occur for $x \to 0$.

Going back to it, (assuming $f(x) = f_0(x) + \varepsilon f_1(x) + \dots$)

we set

$$f_0(x) = e^{-x}$$

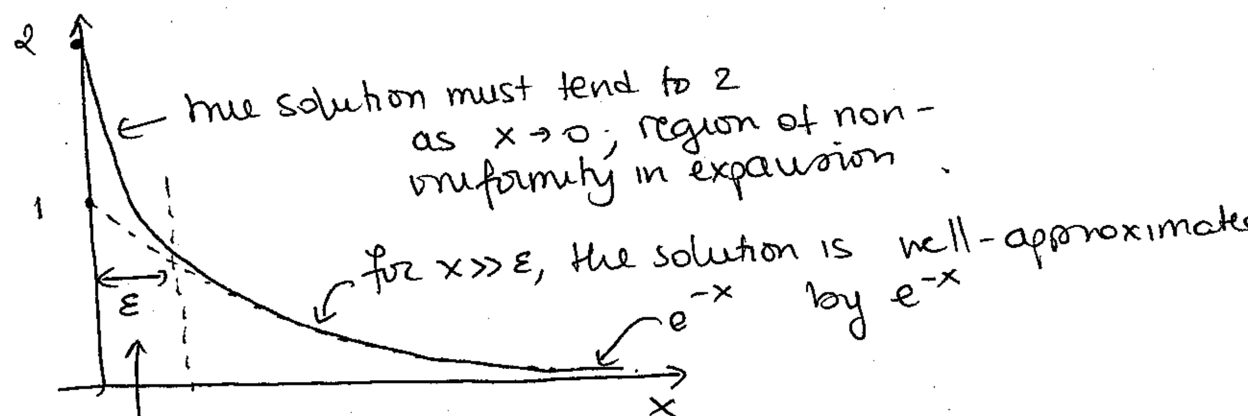
$$f_1(x) = e^{-x}$$

$$\frac{df_1}{dx} + f_2 = 0 \Rightarrow f_2 = -\frac{df_1}{dx} = e^{-x}$$

etc ... and indeed,

we then recover

$$f(x;\varepsilon) \simeq (1+\varepsilon+\varepsilon^2+\dots)\,e^{-x}$$ for large $x$ (that is, $x \gg \varepsilon$)




but clearly, something else must be happening for $x \ll \varepsilon$ to match that solution to the ICs. This bit is captured (in the true solution) by the $e^{-x/\varepsilon}$ part, but can't be done with a simple asymptotic sequence

What happens as $x \to 0$ $(x < \varepsilon)$ is called a "boundary layer". The latter typically occur in ODES as the small parameter multiplies the highest order derivative.

Here, when $\varepsilon = 0$, the equation is <u>algebraic</u> so its not surprising we can't fit the solution to arbitrary boundary conditions. When $\varepsilon \neq 0$, however, we can – but that leads to the emergence of boundary layers, i.e, regions where the $\varepsilon \frac{df}{dx}$ term <u>is</u> important (even if it is multiplied by $\varepsilon$). This requires $f$ to vary <u>rapidly</u> in the boundary layer.

In the next Chapters, we will learn to deal with boundary layers.