I.3 Approximations of functions

So far we have learned how to approximate solutions of ODEs, and solutions of polynomials, i.e quantities defined by a complex equation.

In fact it often happens that we may want to do something much simpler involving evaluating a function at a point $x=a$, or near a point $x=a$.

Example 1: suppose we want to evaluate $e^{0.1}$ but we can't get hold of a calculator.

→ Recall the expansion of e^x near $x=a=0$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots \quad \text{for small } x.$$

$$\rightarrow e^{0.1} = 1 + 0.1 + \frac{0.01}{2} + \frac{0.001}{6} + \cdots$$

$$= 1 + 0.1 + 0.005 + 0.00016\ldots \cong 1.105166\ldots$$

(The exact value is $1.105171\ldots$)

Similar calculations can be used to obtain approximate values of $f(x)$ near a point $a \neq 0$:

Example 2 What is $\sqrt{37}$ (approximately?) without using $\sqrt{\ }$ function?

Since $\sqrt{37} = \sqrt{36+1}$

and $\sqrt{a+\epsilon} = \sqrt{a}\left(1+\frac{\epsilon}{a}\right)^{1/2} \cong \sqrt{a}\left(1 + \frac{1}{2}\frac{\epsilon}{a} + \frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{2}\frac{\epsilon^2}{a^2} + \cdots\right)$

then $\sqrt{37} = \sqrt{36}\left(1 + \frac{1}{2}\frac{1}{36} - \frac{1}{8}\left(\frac{1}{36}\right)^2 + \cdots\right)$

$$= 6\left(1 + \frac{1}{72} - \frac{1}{8(36)^2}\cdots\right) \simeq 6.082\ldots \quad (\text{as in } \sqrt{37})$$

Example 3 Taylor expansions can also be used to approximate functions defined by integrals, such as the error function:

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$

$$\rightarrow \quad \mathrm{erf}(\epsilon) = \mathrm{erf}(0) + \epsilon \left.\frac{d\,\mathrm{erf}}{dx}\right|_0 + \frac{\epsilon^2}{2}\left.\frac{d^2\mathrm{erf}}{dx^2}\right|_0 + \frac{\epsilon^3}{6}\frac{d^3\mathrm{erf}}{dx^3} + \cdots$$

$$= 0 + \epsilon\cdot\frac{2}{\sqrt{\pi}}\left[e^{-t^2}\right]_0 + \frac{\epsilon^2}{2}\cdot\frac{2}{\sqrt{\pi}}\left[-2te^{-t^2}\right]_0$$

$$+ \frac{\epsilon^3}{6}\frac{2}{\sqrt{\pi}}\left[-2e^{-t^2} + 4t^2e^{-t^2}\right]_0 + \cdots$$

$$= \frac{2}{\sqrt{\pi}}\left(\epsilon - \frac{\epsilon^3}{3} + \cdots\right)$$

This method, however, rapidly becomes quite painful. Note, however, since ϵ is small, any t in the interval $[0,\epsilon]$ is <u>also</u> small so we can expand the integrand instead:

$$\rightarrow \quad \mathrm{erf}(\epsilon) = \frac{2}{\sqrt{\pi}}\int_0^\epsilon e^{-t^2}\,dt \simeq \frac{2}{\sqrt{\pi}}\int_0^\epsilon \left(1 - t^2 + \frac{1}{2}t^4 - \frac{1}{6}t^6 + \cdots\right)dt$$

$$= \frac{2}{\sqrt{\pi}}\left(\epsilon - \frac{\epsilon^3}{3} + \frac{\epsilon^5}{10} - \frac{\epsilon^7}{42} + \cdots\right)$$

much more efficient!

Example 4 In all cases above, we were interested in expansions near $x=a$ where a is finite. However it is also possible to get an expansion of a function near ∞, simply by letting $\epsilon = \frac{1}{x}$ and rewriting the function appropriately.

- For instance, we know that $\lim_{x\to+\infty} \frac{1+x}{1-2x} = -\frac{1}{2}$ but how does this function approach ∞?

If $x = \frac{1}{\epsilon}$ then $f(\frac{1}{\epsilon}) = \dfrac{1+\frac{1}{\epsilon}}{1-\frac{2}{\epsilon}} = \dfrac{\epsilon+1}{\epsilon-2} = g(\epsilon)$

If ϵ is small then

$$g(\epsilon) = \frac{(\epsilon+1)}{-2}\frac{1}{(1-\frac{\epsilon}{2})} \simeq \frac{\epsilon+1}{-2}\left(1+\frac{\epsilon}{2}+\frac{\epsilon^2}{4}+\cdots\right)$$

$$\simeq -\frac{1}{2} + \epsilon\left(-\frac{1}{2}-\frac{1}{4}\right) + \epsilon^2\left(-\frac{1}{4}-\frac{1}{8}\right)+\cdots$$

$$\simeq -\frac{1}{2} - \frac{3\epsilon}{4} - \frac{3\epsilon^2}{8} + \cdots$$

which implies in return that

$$f(x) \simeq -\frac{1}{2} - \frac{3}{4x} - \frac{3}{8x^2}$$ when $x \to +\infty$.

- We can try to do the same for the error function to get an estimate of its behaviour as $x\to+\infty$.

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt = \frac{2}{\sqrt{\pi}}\left[\int_0^\infty e^{-t^2}\,dt - \int_x^\infty e^{-t^2}\,dt\right].$$

$$= \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}\,dt - \mathrm{erfc}(x)$$

complementary erf.

Unfortunately, erfc(x) doesn't lend itself to simple expansions as $x \to +\infty$.

Eg. If we try $g(\epsilon) = \text{erfc}\left(\frac{1}{\epsilon}\right) = \frac{2}{\sqrt{\pi}} \int_{\frac{1}{\epsilon}}^{\infty} e^{-t^2}\, dt$

Change $u = \frac{1}{t}$ $\qquad = \frac{2}{\sqrt{\pi}} \int_0^{\epsilon} e^{-\frac{1}{u^2}} \frac{du}{u^2}$

This term does not have a regular taylor expansion near 0.

On the other hand here we can use a trick:

$$\frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \int_x^{\infty} \frac{-2te^{-t^2}}{-2t} dt = \frac{2}{\sqrt{\pi}} \left\{ \left[\frac{e^{-t^2}}{-2t} \right]_x^{\infty} - \int_x^{\infty} \frac{e^{-t^2}}{2t^2} dt \right\}$$

IBP

$$= \frac{2}{\sqrt{\pi}} \left\{ \frac{e^{-x^2}}{2x} - \frac{1}{2} \int_x^{\infty} \frac{e^{-t^2}}{t^2} dt \right\}$$

use same trick

$$= \frac{2}{\sqrt{\pi}} \left\{ \frac{e^{-x^2}}{2x} + \frac{1}{2} \int_x^{\infty} \frac{-2te^{-t^2}}{2t^3} dt \right\}$$

IBP

$$= \frac{2}{\sqrt{\pi}} \left\{ \frac{e^{-x^2}}{2x} + \frac{1}{2} \left[\frac{e^{-t^2}}{2t^3} \right]_x^{\infty} + \frac{3}{4} \int_x^{\infty} \frac{e^{-t^2}}{t^4} dt \right\}$$

$$= \frac{2}{\sqrt{\pi}} e^{-x^2} \left\{ \frac{1}{2x} - \frac{1}{4x^3} + \dots \right\}$$

So finally, near $x = \infty$,

$$\text{erf}(x) \simeq 1 - \frac{e^{-x^2}}{\sqrt{\pi}} \left\{ \frac{1}{x} - \frac{1}{2x^3} + \dots \right\}.$$

CHAPTER 2 Asymptotics

In this Chapter we now introduce some of the more formal definitions & tools that will be needed in this Perturbation methods course.

II.1 Order symbol O (big O)

- Recall the definition of limits from calculus:

If $\underbrace{\lim_{x \to x_0-} f(x)}_{\text{limit approaching } x_0 \text{ from the left}} = \underbrace{\lim_{x \to x_0+} f(x)}_{\text{limit approaching from right}} = f(x_0)$

then we say that

① the limit $\lim_{x \to x_0} f(x)$ exists

② it is equal to $f(x_0)$

③ and the function $f(x)$ is continuous at $x = x_0$.

- Limits at ∞ can also be discussed in similar terms, by remembering how to perform a change of variables:

$$\lim_{x \to +\infty} f(x) = \lim_{y \to 0+} f\left(\frac{1}{y}\right)$$

$$\lim_{x \to -\infty} f(x) = \lim_{y \to 0-} f\left(\frac{1}{y}\right)$$

In this case, however, we only consider 1-sided limits.

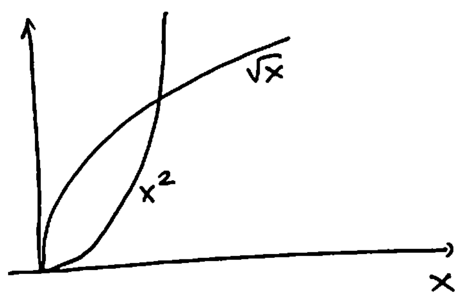
- Finally, note that we can <u>also</u> recast a limit at x_0 into a limit at 0 by another change of variables:

$$\lim_{x \to x_0} f(x) = \lim_{y \to 0} f(y + x_0)$$

In other words we can always treat a limit problem as one in which the independent variable tends to 0.

We now want to answer the question "How fast is $f(x)$ approaching its limit as $x \to 0_+$ or $x \to 0_-$?".

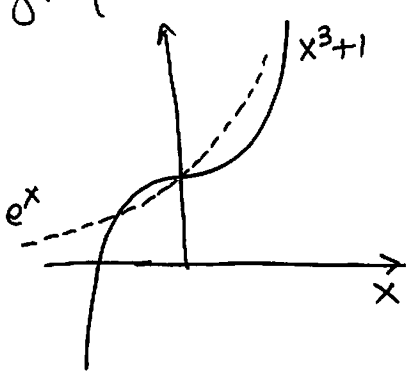
For instance, consider $f(x) = \sqrt{x}$ and $f(x) = x^2$



→ quite clear that x^2 goes to 0 much faster than $\sqrt{x}$

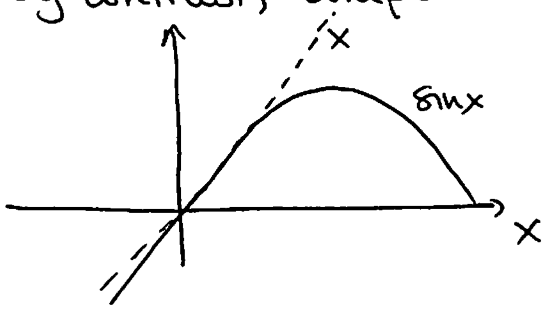
But how do we quantify / formalize this idea?

Similarly, consider $f(x) = x^3 + 1$ and $f(x) = e^x$
In both cases, $\lim_{x \to 0} f(x) = 1$. However, from the graphs we see that



→ here x^3+1 gets to 1 much faster than e^x does.

By contrast, compare $f(x) = x$ and $f(x) = \sin x$



→ sin x and x approach x at the same rate.

The concept of "how rapidly" something approaches 0 (or another limit) can be formalized with the order symbol O (big O). We say that

if $\lim_{x \to 0} \frac{f(x)}{g(x)} = C$

C a constant that is NOT 0

then $f(x) = O(g(x))$ as $x \to 0$.

and say "$f(x)$ is of the order of $g(x)$ as x tends to 0".

Example: $\lim_{x \to 0} \frac{1-\cos x}{x} = 0 \to$ $1-\cos x$ is NOT of the order of x

but $\lim_{x \to 0} \frac{1-\cos x}{x^2} = \frac{1}{2} \to$ $1-\cos x$ is of the order of x^2 as $x \to 0$.

Similarly we can do the same for limits at $x = x_0$ or limits at ∞:

Example: $\lim_{x \to +\infty} \frac{x^2+2x+1}{3x^2} = \frac{1}{3}$ so

x^2+2x+1 is of the order of $3x^2$ as $x \to +\infty$.

$\lim_{x \to \frac{\pi}{2}} \frac{\sin(x-\frac{\pi}{2})}{x-\frac{\pi}{2}} = 1$ so

$\sin(x-\frac{\pi}{2})$ is of the order of $x-\frac{\pi}{2}$ as $x \to \frac{\pi}{2}$.

Note: In the expression $f(x) = O(g(x))$ $g(x)$ is called the gauge function. Usually $g(x)$ is taken to be a power of x; however, this is not always the case.

Since there is no power of x that goes to ∞ faster or as fast as e^x (or any exponential), we must allow for e^x to be a gauge function as well.

Similarly, since there is no power of x (negative power) that go to ∞ more slowly than $\log x$ as $x \to 0$, we also must allow for $\log x$ to be a gauge function.

→ In summary, we see that we will need to consider any power of x, any $\log(x)$, any e^x and any product thereof as gauge functions, at the very least.

II.2 Taylor & McLaurin series with remainder

① Taylor series (infinite series)

Suppose that $f(x)$ is infinitely differentiable, then for x near enough x_0

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-x_0)^n}{n!} \left.\frac{d^n f}{dx^n}\right|_{x=x_0}$$

② Taylor's theorem for series with remainder

Suppose that $f(x)$ is differentiable at least $N+1$ times then for x near enough x_0

$$f(x) = \sum_{n=0}^{N} \frac{(x-x_0)^n}{n!} \left.\frac{d^n f}{dx^n}\right|_{x=x_0} + \underbrace{R_N}_{\text{remainder}}$$

where $R_N = \dfrac{(x-x_0)^{N+1}}{(N+1)!} \left.\dfrac{d^{N+1} f}{dx^{N+1}}\right|_{x=a}$

where a is a point lying between x and x_0.

Proof:

Note that $f(x) = f(x_0) + \int_{x_0}^{x} f'(t)\,dt$

then integrate the second term by parts:

$$f(x) = f(x_0) + \Big[(t-x)f'(t)\Big]_{x_0}^{x} + \int_{x_0}^{x} f''(t)(x-t)\,dt$$

IBP again

$$= f(x_0) + (x-x_0)f'(x_0) + \Big[-\frac{(t-x)^2}{2}f''(t)\Big]_{x_0}^{x} + \int_{x_0}^{x} f'''(t)\frac{(x-t)^2}{2}\,dt$$

$$= f(x_0) + (x-x_0)f'(x_0) + \frac{(x-x_0)^2}{2}f''(x_0) + \dots \text{ etc.}$$

} after N iterations

$$= \sum_{n=0}^{N} \frac{(x-x_0)^n}{n!} f^{(n)}(x_0) + \int_{x_0}^{x} f^{(N+1)}(t)\frac{(x-t)^N}{N!}\,dt$$

Using mean value theorem + more calculus, we can then show that

$$\int_{x_0}^{x} f^{(N+1)}(t)\frac{(x-t)^N}{N!}\,dt = f^{(N+1)}(a)\frac{(x-x_0)^{N+1}}{(N+1)!}$$

where a is some point between x and x_0.

Notes: • If we consider $x_0 = 0$, then we get the McLaurin formula

$$f(x) = \sum_{n=0}^{N} \frac{x^n f^{(n)}(0)}{n!} + R_N \quad \text{where } R_N = \frac{x^{N+1}}{(N+1)!} f^{(N+1)}(a)$$

for $0 < a < x$

• Having assumed that the N+1 derivative exists and is continuous, we can then say that

$$\lim_{x \to x_0} \frac{R_N}{(x-x_0)^{N+1}} = \frac{f^{(N+1)}(a)}{(N+1)!} = \text{a constant}$$

→ this then says that $R_N = O\left((x-x_0)^{N+1}\right)$

so we can write $f(x) = \sum_{n=0}^{N} \frac{(x-x_0)^n}{n!} \left.\frac{d^n f}{dx^n}\right|_{x_0} + O\left((x-x_0)^{N+1}\right)$

Examples of famous series

as $x \to 0$:
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$
$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$
$$(1+x)^a = 1 + ax + \frac{a(a-1)}{2!} x^2 + \frac{a(a-1)(a-2)}{3!} x^3 + \cdots$$

③ Two views of the Taylor Series:

Consider a function that is infinitely differentiable, and it's Taylor series with remainder near x_0

$$f(x) = \sum_{n=0}^{N} \frac{(x-x_0)^n}{n!} \left.\frac{d^n f}{dx^n}\right|_{x_0} + \underbrace{\frac{(x-x_0)^{N+1}}{(N+1)!} \left.\frac{d^{N+1} f}{dx^{N+1}}\right|_{x=a}}_{R_N}$$

Because we have shown that $R_N = o(x-x_0)^{N+1}$ we know that

$$\lim_{x \to x_0} f(x) - \sum_{n=0}^{N} \frac{(x-x_0)^n}{n!} \left.\frac{d^n f}{dx^n}\right|_{x_0} = 0$$

→ in otherwords, for fixed N ,

the finite sum becomes a better & better approximation to $f(x)$ as $x \to x_0$

However, the "symmetric limit" for $N \to \infty$ at fixed x is much less obvious.

7. Indeed, if we consider a value of $x \neq x_0$ (but close to it

$$\lim_{N \to +\infty} \left[f(x) - \sum_{n=0}^{N} \frac{(x-x_0)^n}{n!} \frac{d^n f}{dx^n}\Big|_{x_0} \right]$$

This limit is only equal to 0 iff

$$\lim_{N \to +\infty} \frac{(x-x_0)^{N+1}}{(N+1)!} \frac{d^{N+1} f}{dx^{N+1}}\Big|_{x=a} = 0$$

→ This is a lot harder to prove, usually. Not all infinitely differentiable functions satisfy this.

<u>Definition</u>: if a function $f(x)$, infinitely differentiable, satisfies

$$\lim_{N \to \infty} \frac{(x-x_0)^N}{N!} \frac{d^N f}{dx^N}\Big|_{x=a} = 0$$

for x in a <u>given</u> neighborhood D of x_0, and a is any point within D, then $f(x)$ is said to be <u>analytic</u> in D.

<u>Property of analytic functions</u>:

> The Taylor series of an analytic function is convergent for any x in D

This means that for any x in D, we <u>can</u> get progressively <u>better</u> approximations to $f(x)$ by keeping more terms in the series.

This property, as we just saw, is non-trivial, and

- there are some functions for which this is never true (non-analytic functions)
- for x outside of D, the Taylor series does <u>not</u> converge as $N \to +\infty$ (meaning that the remainder does <u>NOT</u> tend to 0). Hence the qualifier, "for x close enough to x_0".