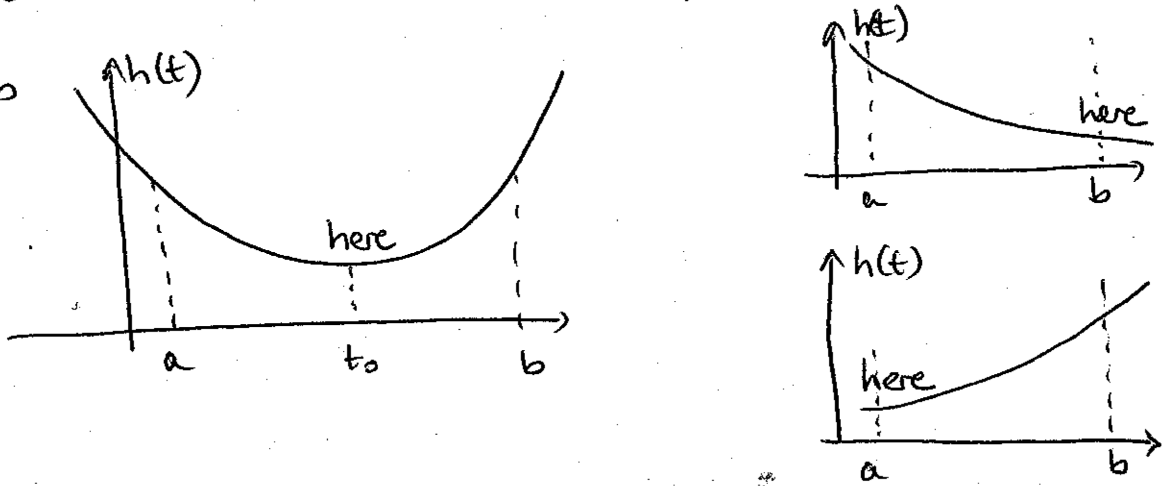
III Laplace's method

In the previous section, we saw first examples of what to do in the more general case where
$$I = \int_a^b e^{-\lambda h(t)} f(t)\,dt,$$
<u>provided $h(t)$ is one-to-one</u>. In that case a "simple" change of variables reduces the problem to one that can be dealt with Watson's lemma.

However, this doesn't always work. First, the change of variable may not always lead to simple algebra. Secondly, this method can't work of course if $h(t)$ is <u>not</u> one-to-one, ie if $h(t)$ has a minimum or a maximum in the interval $[a,b]$. When this is the case, we apply Laplaces' method, which is essentially a generalization of the method that lead us to derive Watson's Lemma in the first place, That is, the region of dominant contribution.

If $\lambda > 0$, then the region of dominant contribution is wherever $h(t)$ is smallest

<u>Examples</u>



10.

<u>Case 1</u> Near an interior point where $h'(t_0)=0$ and $h''(t_0)>0$

→ As before we expand $f(t)$ (and now also $h(t)$) near $t=t_0$:

$$f(t) = f(t_0) + (t-t_0) f'(t_0) + \dots$$

$$h(t) = h(t_0) + \frac{(t-t_0)^2}{2} h''(t_0) + \dots$$

$$\to \quad I = \int_a^b e^{-\lambda h(t)} f(t)\, dt \simeq \int_a^b e^{-\lambda\left(h(t_0) + \frac{(t-t_0)^2}{2} h''(t_0) + \dots\right)} \cdot \left(f(t_0) + (t-t_0) f' \dots\right)$$

$$= e^{-\lambda h(t_0)} \int_a^b e^{-\lambda h''(t_0)\frac{(t-t_0)^2}{2} + \dots} \left(f(t_0) + (t-t_0) f'(t_0) + \dots \right) dt$$

let $x^2 = \lambda h''(t_0) \dfrac{(t-t_0)^2}{2}$ → $x = \sqrt{\dfrac{\lambda h''(t_0)}{2}}\,(t-t_0)$

$$\text{So} \quad I = e^{-\lambda h(t_0)} \int_{(a-t_0)\sqrt{\frac{\lambda h''}{2}}}^{(b-t_0)\sqrt{\frac{\lambda h''}{2}}} \sqrt{\frac{2}{\lambda h''}}\, e^{-x^2 + \dots} \left(f(t_0) + \sqrt{\frac{2}{\lambda h''}}\, x f'(t_0) \dots \right) dx$$

let's look at this term by term.

$$\text{let } I_1 = \int_{(a-t_0)\sqrt{\frac{\lambda h''}{2}}}^{(b-t_0)\sqrt{\frac{\lambda h''}{2}}} e^{-x^2}\, dx = \int_{(a-t_0)\sqrt{\frac{\lambda h''}{2}}}^{-\infty} e^{-x^2}\, dx + \int_{-\infty}^{+\infty} e^{-x^2}\, dx + \int_{+\infty}^{(b-t_0)\sqrt{\frac{\lambda h''}{2}}} e^{-x^2}\, dx$$

$$= \int_{(t_0-a)\sqrt{\frac{\lambda h''}{2}}}^{+\infty} e^{-x^2}\, dx + \sqrt{\pi} - \int_{(b-t_0)\sqrt{\frac{\lambda h''}{2}}}^{+\infty} e^{-x^2}\, dx$$

$$= \frac{\sqrt{\pi}}{2} \operatorname{erfc}\Big(\underbrace{(t_0-a)\sqrt{\tfrac{\lambda h''}{2}}}_{\Lambda_a}\Big) + \sqrt{\pi} - \frac{\sqrt{\pi}}{2} \operatorname{erfc}\Big(\underbrace{(b-t_0)\sqrt{\tfrac{\lambda h''}{2}}}_{\Lambda_b}\Big)$$

The two erfc functions, as we saw last time, are well-approximated by $\operatorname{erfc}(\Lambda) = \dfrac{2}{\sqrt{\pi}} e^{-\Lambda^2} \cdot \dfrac{1}{2\Lambda}$ for $\Lambda \gg 1$.

so $I_1 = \sqrt{\pi}$ + exponentially small terms!

let $I_2 = \displaystyle\int_{(a-t_0)\sqrt{\frac{\lambda h''}{2}}}^{(b-t_0)\sqrt{\frac{\lambda h''}{2}}} x e^{-x^2}\,dx$ let $u = x^2$

$$= \int_{(t_0-a)^2\frac{\lambda h''}{2}}^{(b-t_0)^2\frac{\lambda h''}{2}} \frac{1}{2} e^{-u}\,du$$

$$= \frac{1}{2}\left[e^{-(t_0-a)^2\frac{\lambda h''}{2}} - e^{-(t_0-b)^2\frac{\lambda h''}{2}} \right]$$

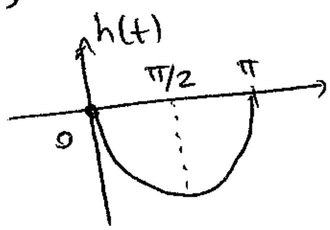
again, exponentially small terms in λ.

$\rightarrow$ to a first approximation (neglecting exponentially-small terms)

$$I = \int_a^b e^{-\lambda h(t)} f(t)\,dt = f(t_0) e^{-\lambda h(t_0)} \sqrt{\frac{2\pi}{\lambda h''(t_0)}} \quad \text{for} \begin{cases} \lambda > 0 \\ h''(t_0) > 0 \\ a < t_0 < b. \end{cases}$$

as $\lambda \to +\infty$

<u>Example</u>: $\displaystyle\int_0^{\pi} e^{\lambda \sin t}\,dt \quad (\lambda > 0)$



$\rightarrow t_0 = \frac{\pi}{2}$

$f(t) = 1$, $h(t) = -\sin t$ here.

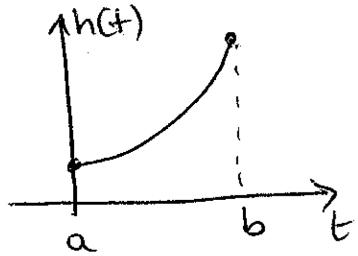
$\rightarrow h''(t_0) = \sin t_0 = 1$

$$\Rightarrow \int_0^{\pi} e^{\lambda \sin t}\,dt \simeq e^{\lambda} \cdot \sqrt{\frac{2\pi}{\lambda}} + \text{h.o.t.}$$

<u>Note</u>: If we actually want the h.o.t's we need to work to higher order in the expansion of $h(t)$.

Case 2 : If the region of dominant contribution is near one of the boundaries, and $h'(t) \neq 0$ there, we can reduce the problem to Watson's Lemma

Case 3 : If the region of dominant contribution is on the boundaries and $h'(t) = 0$ there, then we proceed as in case 1. Suppose its at $x = a$:



$\Rightarrow \quad h''(a) > 0.$
$\quad\quad h'(a) = 0$

$\rightarrow \quad h(t) = h(a) + \frac{(t-a)^2}{2} h''(a) + \ldots$

so

$$I = \int_a^b e^{-\lambda h(t)} f(t)\,dt \simeq \int_a^b e^{-\lambda\left(h(a) + \frac{(t-a)^2}{2} h''(a) + \cdots\right)} \left(f(a) + (t-a) f'(a) \ldots\right.$$

$$= e^{-\lambda h(a)} \int_a^b e^{-\lambda \frac{(t-a)^2}{2} h''(a)} \left(f(a) + (t-a) f'(a) + \ldots\right) dt$$

$$\simeq e^{-\lambda h(a)} \int_0^{\frac{(b-a)^2 \lambda h''}{2}} e^{-x^2} \cdot \sqrt{\frac{2}{\lambda h''}}\, f(a)\,dx \quad + \text{hot}$$

$$\simeq e^{-\lambda h(a)} f(a) \sqrt{\frac{2}{\lambda h''(a)}} \int_0^\infty e^{-x^2}\,dx \quad + \text{hot and exponentially small terms}$$

$$\simeq e^{-\lambda h(a)} f(a) \sqrt{\frac{\pi}{2\lambda h''(a)}} \quad + \text{h.o.t}$$

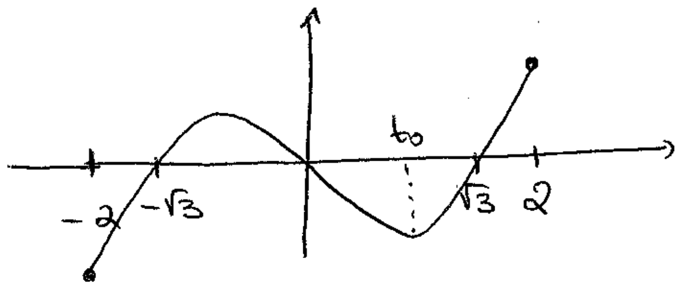
... Etc $\rightarrow$ we can continue working through many different cases, each time applying a similar method.

The only "somewhat" tricky remaining issue is what to do when there are multiple regions of dominant contribution (in the case $\lambda>0$, multiple minima of h, both in the domain and/or on the boundary).

<u>Case 4</u>: Multiple minima. (Examples).

$$I = \int_{-2}^{2} e^{-\lambda\left(\frac{t^3}{3}-t\right)} \ln(t^2+1)\, dt \quad \text{as } \lambda \to +\infty .$$

$$h(t) = \frac{t^3}{3} - t = t\left(\frac{t^2}{3}-1\right) = \frac{1}{3}t\,(t-\sqrt{3})(t+\sqrt{3})$$



$h(t)$ has 2 minima in $[-2,2]$

- one at $t=-2$
- one at $t=t_0$ interior to interval.

to find t_0, $h'(t) = t^2-1 \Rightarrow h'(t)=0 \Rightarrow t_0 = \pm 1$

$\rightarrow$ we clearly want the $t_0=1$ one.

To deal with multiple minima, we simply chop the interval into parts that contain only 1 of the original minima each:

$$I = \int_{-2}^{0} \cdots + \int_{0}^{2} \cdots = I_1 + I_2$$

The fact that we "introduce" another "boundary minimum" at 0 doesn't matter since it will eventually cancel out when summing I_1 & I_2.

$\rightarrow$ In I_1, the region of dominant contribution is $t=-2$, and in I_2, it is $t=1$.

From our previous work we already know that

$$I_2 \simeq e^{-\lambda h(1)} \cdot f(1) \cdot \sqrt{\frac{2\pi}{\lambda h''(1)}} \qquad h(1) = \tfrac{1}{3} - 1 = -\tfrac{2}{3}$$

$$h''(1) = 2$$

$$= e^{\frac{2}{3}\lambda} \ln 2 \sqrt{\frac{\cancel{2}\pi}{\cancel{2}\lambda}}. \qquad f(1) = \ln(2)$$

Meanwhile, expanding h and f near $t=-2$ we get

$$I_1 \cong \int_{-2}^{0} e^{-\lambda h(-2) - \lambda(t+2)h'(-2) + \cdots} \left(f(-2) + (t+2) f'(-2) + \cdots \right) dt$$

$$h(-2) = \frac{-8}{3} + 2 = -\frac{2}{3} \qquad h'(-2) = 4 - 1 = 3 \qquad s$$

$$f(-2) = \ln(5) \qquad f'(-2) = \frac{-4}{5} \quad \cdots$$

since $f'(t) = \frac{2t}{t^2+1}$

$$\rightarrow I_1 \simeq e^{\frac{2}{3}\lambda} \int_{-2}^{0} e^{-3\lambda(t+2)} f(t)\, dt$$

let $s = 3(t+2)$ then $t = \frac{s}{3} - 2$

$$I_1 = \frac{e^{\frac{2}{3}\lambda}}{3} \underbrace{\int_0^6 e^{-s\lambda} f\left(\tfrac{s}{3} - 2\right) ds}_{\text{This leads to direct application of Watson's Lemma}}$$

$$= \frac{e^{\frac{2}{3}\lambda}}{3} \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{\lambda^{n+1}} \qquad \text{where } g(s) = f\left(\tfrac{s}{3} - 2\right)$$

$$= \frac{e^{\frac{2}{3}\lambda}}{3} \left(\frac{\ln 5}{\lambda} - \frac{4}{5} \cdot \frac{1}{\lambda^2} + \cdots \right)$$

$\longrightarrow$ To lowest degree of approximation (neglecting I_1 in front of I_2 since $I_1 = o(I_2)$ as $\lambda \to +\infty$)

we get $\displaystyle\int_{-2}^{2} \exp\left(-\lambda\left(\tfrac{t^3}{3} - t\right)\right) \ln(1+t^2)\, dt \simeq e^{\frac{2}{3}\lambda} \ln 2 \left(\frac{\pi}{\lambda}\right)^{\frac{1}{2}}$

IV The Riemann-Lebesgue Lemma

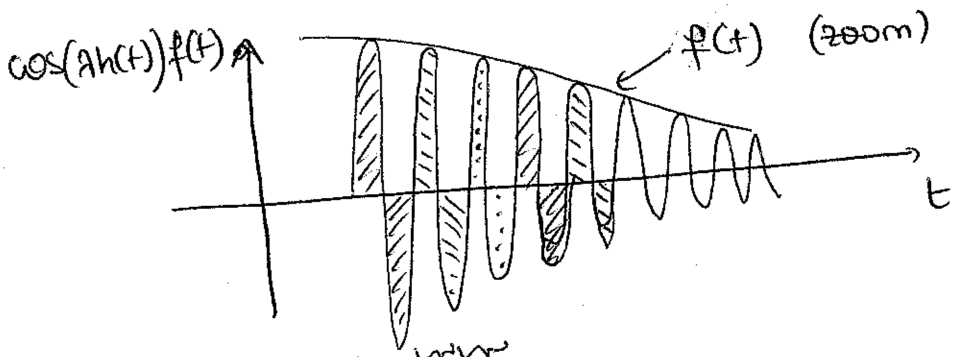
So far we have considered integrals of the form

$$\int_a^b e^{-\lambda h(t)} f(t)\, dt.$$

Let's now turn to the other ones introduced originally:

$$\int_a^b \cos(\lambda h(t)) f(t)\, dt \quad \& \quad \int_a^b \sin(\lambda h(t)) f(t)\, dt.$$

Note how if λ is very large, then the functions $\cos(\lambda h(t))$ and $\sin(\lambda h(t))$ typically oscillate very rapidly – much more rapidly than the scale over which $f(t)$ varies



$\Rightarrow$ integral over successive pairs of peaks & troughs nearly cancel out!

As a result, under certain conditions, we expect that the integrals $\int_a^b \cos(\lambda h(t)) f(t)\, dt$ & $\int_a^b \sin(\lambda h(t)) f(t) dt$ should go to 0 as $\lambda \to \infty$. This is in fact what the Riemann-Lebesgue Lemma states.

<u>Lemma</u>: Suppose $\int_a^b |f(t)| dt < +\infty$ then

$$\lim_{\lambda \to +\infty} \int_a^b \cos(\lambda h(t)) f(t)\, dt = \lim_{\lambda \to +\infty} \int_a^b \sin(\lambda h(t)) f(t) dt$$

$$= 0$$

Note that a and/or b can be $\pm\infty$

The condition $\int_a^b |f(t)|\,dt$ simply states that the integral exists & is finite. Examples where this is not the case would be $\int_0^b \frac{1}{t}\,dt$, $\int_0^\infty e^t\,dt$, etc...

The proof of the lemma is fairly fiddly so we will not study it here (it simply involves chopping the interval into chunks of length $\frac{2\pi}{\lambda h(t)}$ & studying the limit of the sum as $\lambda \to \infty$ ...).

Let's look at examples of application / verification

- $I = \int_a^b \cos \lambda t\,dt = \left[\frac{1}{\lambda}\sin(\lambda t)\right]_a^b \longrightarrow 0$ as $\lambda \to +\infty$

- $I = \int_a^b t \sin \lambda t\,dt = \left[\frac{\sin(\lambda t) - \lambda t \cos \lambda t}{\lambda^2}\right]_a^b \longrightarrow 0$ as $\lambda \to +\infty$

  In both cases, as long as a & b are finite we clearly satisfy $\int_a^b |f(t)|\,dt < +\infty$.

- Let's now look at

  $$I = \int_0^1 \frac{\sin \lambda t}{t}\,dt$$

  In this case, we can't use the lemma because

  $$\int_0^1 \left|\frac{1}{t}\right| dt$$

  is unbounded.

  However, we know that this integral must exist, since $\frac{\sin \lambda t}{t} \longrightarrow \lambda$ as $t \to 0$ (so the lower limit of the integral is well-defined).

The problem with this lemma is that it doesn't tell 17.
us much
- doesn't state how fast $I \to 0$ as $\lambda \to +\infty$ even when Lemma applies
- leaves us in the dark if lemma doesn't apply even if we know the integral exists

To find a good asymptotic expansion often requires tricks (e.g. integration by parts, etc) & a lot of trial & error.

A "simple" case

$$\int_1^2 \frac{1}{\sqrt{t}} \cos\lambda t \, dt$$ → here we'll use IBP smartly

$$= \left[\frac{\sin\lambda t}{\lambda\sqrt{t}} \right]_1^2 + \frac{1}{2}\int_1^2 \frac{\sin\lambda t}{\lambda t^{3/2}}\,dt$$ ← Note how doing it this way around we get a $\frac{1}{\lambda}$ ...

$$= \frac{\sin 2\lambda}{\sqrt{2}\,\lambda} - \frac{\sin\lambda}{\lambda} + \frac{1}{2}\left[\frac{-\cos\lambda t}{\lambda^2 t^{3/2}} \right]_1^2 + \frac{1}{2}\int_1^2 \frac{\cos\lambda t}{\lambda^2}\cdot\left(-\frac{3}{2}\right)t^{-5/2}\,dt$$

$$= \frac{1}{\lambda}\left(\frac{\sin 2\lambda}{\sqrt{2}} - \sin\lambda \right) + \frac{1}{2\lambda^2}\left(\cos\lambda - \frac{\cos 2\lambda}{2\sqrt{2}} \right) + O\left(\frac{1}{\lambda^3}\right)$$

→ all good!

A "tricky case" → looks similar, but much harder.

$$\int_0^1 \frac{1}{\sqrt{t}} \cos\lambda t \, dt$$ → again Lemma applies. So we expect this → 0.

$$= \left[\frac{\sin\lambda t}{\lambda\sqrt{t}} \right]_0^1 + \frac{1}{2}\int_0^1 \frac{\sin\lambda t}{\lambda t^{3/2}}\,dt$$

$$= \frac{\sin\lambda}{\lambda} - 0 + \frac{1}{2}\left[\frac{-\cos\lambda t}{\lambda^2 t^{3/2}} \right]_0^1 + \underbrace{\frac{1}{2}\int_0^1 \frac{\cos\lambda t}{\lambda^2}\left(-\frac{3}{2}\right)t^{-5/2}\,dt}_{\text{this integral doesn't exist...}}$$

??

lower limit does not exist

→ does this still mean that $I \sim \frac{\sin\lambda}{\lambda}$??

If we do it the other way then we get

$$\int_0^1 \frac{1}{\sqrt{t}} \cos\lambda t \, dt = \left[2\sqrt{t}\cos\lambda t\right]_0^1 + \int_0^1 2\sqrt{t}\,\lambda \sin\lambda t \, dt$$

$$= 2\cos\lambda + \left[2 \cdot \tfrac{2}{3} t^{3/2} \lambda \sin\lambda t\right]_0^1 - \int_0^1 2 \cdot \tfrac{2}{3} t^{3/2} \lambda^2 \cos\lambda t \, dt$$

$$= 2\cos\lambda + \tfrac{4}{3}\lambda \sin\lambda - \dots$$

→ each integral converges, but we get a series in positive powers of λ!! ??

For this problem, the only way forward is to split the integral artificially:

$$\int_0^1 \frac{1}{\sqrt{t}} \cos\lambda t \, dt = \int_0^\infty \frac{1}{\sqrt{t}} \cos\lambda t \, dt - \int_1^\infty \frac{1}{\sqrt{t}} \cos\lambda t \, dt$$

use IBP

↑ use change of variables $u = \sqrt{\lambda t}$

$du = \sqrt{\lambda} \cdot \frac{1}{2\sqrt{t}} \, dt$

$$= \int_0^\infty \frac{2}{\sqrt{\lambda}} \cos u^2 \, du - \left[\frac{\sin\lambda t}{\lambda\sqrt{t}}\right]_1^\infty - \frac{1}{2}\int_1^\infty \frac{\sin\lambda t}{\lambda t^{3/2}} \, dt$$

Wolfram:

$\int_0^\infty \cos x^2 \, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$

$$= \frac{\cancel{2}}{\sqrt{\lambda}} \cdot \frac{1}{\cancel{2}}\sqrt{\frac{\pi}{2}} + \frac{\sin\lambda}{\lambda} - \underbrace{\frac{1}{2\lambda}\int_1^\infty \frac{\sin\lambda t}{t^{3/2}} \, dt}_{\text{continue IBP on this one, to show its } O\left(\frac{1}{\lambda^2}\right) \dots}$$

$$\rightarrow \int_0^1 \frac{1}{\sqrt{t}} \cos\lambda t \, dt = \sqrt{\frac{\pi}{2\lambda}} + O\left(\frac{1}{\lambda}\right)$$

oof!

Note that it is sadly very easy to spend hours trying to get a meaningful answer → key is to have lots of experience!

V The method of stationary phases

The Riemann-Lebesgue lemma, & associated "tricks", were for integrals of the kind $\int_a^b \cos(\lambda t) f(t)\, dt$ (and similarly for a sine integral). Let's now study a more general method for $\int_a^b \cos(\lambda h(t)) f(t)\, dt$ called the <u>method of stationary phases.</u>, which could also be called the method of zero frequency (which I think would be clearer).

- Since the cancellations in successive phases of the oscillation cause the integral to be close to 0, we expect these cancellations to be <u>least effective</u> when the frequency of oscillation is the smallest. (ideally, zero).

→ The region of dominant contribution in integrals of the kind $\int_a^b \cos(\lambda h(t)) f(t)\, dt$ are regions where $\lambda h(t)$ change the least rapidly, i.e. when $\frac{dh}{dt} = 0$.

Another way of seeing this is to let

$$h(t) \simeq h(t_0) + h'(t_0)(t-t_0) + h''(t_0)\frac{(t-t_0)^2}{2} + \ldots$$

$$\to \cos(\lambda h(t_0) + \lambda h'(t_0)(t-t_0) + \ldots)$$

$$= \cos(\varphi_0 + \underbrace{\lambda h'(t_0)}_{\omega_0} t)$$

ω_0 = the local frequency of oscillation

→ zero frequency $\Leftrightarrow h'(t_0) = 0$.

Note that the standard method of stationary phases only considers the case where h actually has a stationary point in interval
→ we shall limit ourselves to this case here too.

The manner in which we proceed is essentially the same as in the exponential case: expand $f(t)$ and $h(t)$ around t_0, & treat solution term by term.

let $$I = \int_a^b \cos(\lambda h(t))\, f(t)\, dt$$

$$= \int_a^b \cos\left(\lambda h(t_0) + \lambda \frac{(t-t_0)^2}{2} h''(t_0) + \dots\right)\left(f(t_0) + (t-t_0)f'(t_0) + \dots\right) dt$$

$$= \cos(\lambda h(t_0)) \cdot \int_a^b \cos\left(\lambda \frac{(t-t_0)^2}{2} h''(t_0) + \dots\right)\left(f(t_0) + (t-t_0)f'(t_0) \dots\right) dt$$

$$- \sin(\lambda h(t_0)) \cdot \int_a^b \sin\left(\lambda \frac{(t-t_0)^2}{2} h''(t_0) + \dots\right)\left(f(t_0) + \dots\right) dt$$

→ so we basically have to evaluate integrals of the form, to lowest order,

$$I_c = \int_a^b \cos\left(\lambda \frac{(t-t_0)^2}{2} h''(t_0)\right) dt$$ and similarly in sine.

let $x^2 = \lambda \frac{(t-t_0)^2}{2} |h''(t_0)| \Rightarrow x = \sqrt{\frac{\lambda}{2}}\,(t-t_0)\sqrt{|h''|}$

$$\Rightarrow I_c = \int_{\sqrt{\frac{\lambda}{2}}(a-t_0)\sqrt{|h''|}}^{\sqrt{\frac{\lambda}{2}}(b-t_0)\sqrt{|h''|}} \cos(x^2)\sqrt{\frac{2}{\lambda |h''|}}\, dx$$

$$= \int_{\sqrt{\frac{\lambda}{2}}(a-t_0)\sqrt{|h''|}}^{-\infty} \dots\, dx + \int_{-\infty}^{+\infty} \dots\, dx + \int_{+\infty}^{\sqrt{\frac{|h''|\lambda}{2}}(b-t_0)} \dots\, dx$$

→ we have 3 integrals to evaluate, one of the form

- $\int_{-\infty}^{+\infty} \cos x^2 \, dx$, which we already know is equal to $2\int_0^{+\infty} \cos x^2 \, dx = \sqrt{\frac{\pi}{2}}$
- two of the form $\int_{\Lambda}^{+\infty} \cos x^2 dx$ where $\Lambda \gg 1$

To evaluate the latter, we use a trick:

$$\int_{\Lambda}^{+\infty} \cos x^2 \, dx = \int_{\Lambda}^{+\infty} \frac{2x}{2x} \cos x^2 \, dx = \left[\frac{\sin x^2}{2x} \right]_{\Lambda}^{+\infty} + \int_{\Lambda}^{+\infty} \frac{\sin x^2}{2x^2} dx$$

$$= \cdot \frac{\sin \Lambda^2}{2\Lambda} + \int_{\Lambda}^{+\infty} \frac{\sin x^2}{2x^2} dx .$$

We can bound the second integral:

$$\left| \int_{\Lambda}^{+\infty} \frac{\sin(x^2)}{2x^2} dx \right| < \int_{\Lambda}^{+\infty} \frac{|\sin(x^2)|}{2x^2} dx < \int_{\Lambda}^{+\infty} \frac{1}{2x^2} dx$$

$$= \left[-\frac{1}{2x} \right]_{\Lambda}^{+\infty} = \frac{1}{2\Lambda}$$

so overall, the contribution from $\int_{\Lambda}^{+\infty} \cos x^2 \, dx = O\left(\frac{1}{\Lambda}\right)$.

→ much smaller than the one from $-\infty$ to $+\infty$

$\Rightarrow$ $I_c \simeq \sqrt{\frac{\cancel{2}}{\lambda |h''(t_0)|}} \cdot \frac{\sqrt{\pi}}{\sqrt{\cancel{2}}}$ and similarly, we can show

that $I_s = \int_a^b \sin\left(\lambda \frac{(t-t_0)^2}{2} h''(t_0)\right) dt = \pm I_c$ to lowest order

where the sign $\pm$ is $+$ if $h''(t_0) > 0$ and $-$ if $h''(t_0) < 0$

$\Rightarrow$ Finally, we get

$$I = \int_a^b \cos(\lambda h(t)) f(t) \, dt$$

$$= f(t_0) \sqrt{\frac{\pi}{\lambda |h''(t_0)|}} \left(\cos(\lambda h(t_0)) \mp \sin(\lambda h(t_0)) \right)$$

and similarly, it's easy to show that

$$I = \int_a^b \sin(\lambda h(t))\, f(t)\, dt$$

$$= f(t_0)\sqrt{\frac{\pi}{\lambda |h''(t_0)|}}\left(\cos(\lambda h(t_0)) \pm \sin(\lambda h(t_0))\right)$$

- Note that if it so happens that $h'(t) = 0$ on either a or b, we proceed again as before, & find approximate solutions that are exactly $\frac{1}{2}$ what they would be if t_0 is an interior point

- If there are multiple stationary points, then as before, we chop the interval in bits which each contain one point.

- Finally, if there are no stationary points at all, then the region of dominant contribution is simply the boundary. We can get approximations to the integral using this trick:

$$I = \int_a^b \cos(\lambda h(t)) f(t)\, dt$$

$$= \int_a^b \lambda h'(t) \cos(\lambda h(t)) \frac{f(t)}{\lambda h'(t)}\, dt$$ ← we can do this since $h'(t) \neq 0$ everywhere.

$$= \left[\sin(\lambda h(t)) \frac{f(t)}{\lambda h'(t)}\right]_a^b - \underbrace{\frac{1}{\lambda}\int_a^b \sin(\lambda h(t)) \cdot \frac{d}{dt}\left(\frac{f}{\lambda h'}\right) dt}_{\text{If Riemann-Lebesgue Lemma applies, then this} \to 0}$$

$$I = \frac{1}{\lambda}\left[\frac{\sin(\lambda h(b)) f(b)}{h'(b)} - \frac{\sin(\lambda h(a)) f(a)}{h'(a)}\right] + o\left(\frac{1}{\lambda}\right).$$