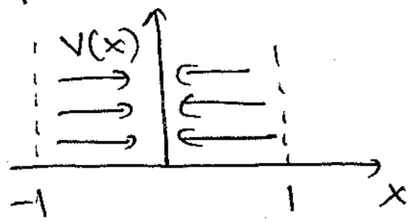
## Example 3: An internal boundary layer.

Consider the equation $\varepsilon \frac{d^2f}{dx^2} + x\frac{df}{dx} + xf = 0$

$$f(-1) = e \qquad f(1) = \frac{2}{e}$$

As before we can interpret this as the end-product of an advection/diffusion process, where the advection velocity is $v(x) = -x$




→ we expect a boundary layer at $x=0$. However, since the actual boundaries are now at $x=1$ and $x=-1$, this is an internal layer.

- Let's look at the outer expansion. This time, "outer" means for $x$ out of the bl, that is, not too small.

let $f = f_0 + \varepsilon f_1 + \cdots$

⇒ to lowest order we have

$$x\frac{df_0}{dx} + xf_0 = 0 \Rightarrow f_0 = ke^{-x}$$

Clearly, we cannot fit both BCs to this $f_0$ function. However, we can create two pieces of the outer expansion as

$$\begin{cases} f_0^+(x) = K^+e^{-x} & \text{for } x > 0 \quad (x \text{ not in BL}) \\ f_0^-(x) = K^-e^{-x} & \text{for } x < 0 \quad (x \text{ not in BL}) \end{cases}$$

$f_0^+$ then satisfies the BC at $x=1 \Rightarrow K^+ = 2$

$f_0^-$ then satisfies the BC at $x=-1 \Rightarrow K^- = 1$

- The inner expansion is determined as usual. First assume that the inner variable is $s = \frac{x}{\varepsilon^p}$, then select $p$ using Van Dyke's principle:

$$\varepsilon \frac{d^2 f}{dx^2} + x \frac{df}{dx} + xf = 0 \Rightarrow \varepsilon^{1-2p} \frac{d^2 f}{ds^2} + s \frac{df}{ds} + \varepsilon^p s f = 0$$

Clearly the 3rd term is much smaller than the second for $p > 0 \Rightarrow$ we then want a balance between the first & second term $\Rightarrow$ take $\varepsilon^{1-2p} = 1 \Rightarrow p = \frac{1}{2}$

The leading term of the inner expansion must then satisfy

$$\frac{d^2 f_0^{in}}{ds^2} + s \frac{df_0^{in}}{ds} = 0$$

$$\Rightarrow \text{let } g = \frac{df_0}{ds} \Rightarrow \frac{dg}{ds} + sg = 0$$

$$\Rightarrow \frac{dg}{g} = -s\,ds \Rightarrow \ln g = -\frac{s^2}{2} + C$$

$$\Rightarrow g = K^{in} e^{-\frac{s^2}{2}}$$

$$\Rightarrow f_0^{in} = \int_0^s K^{in} e^{-\frac{s'^2}{2}} ds' + f_0^{in}(0)$$

$$= \sqrt{\frac{\pi}{2}}\, K_{in} \operatorname{erf}\left(\frac{s}{\sqrt{2}}\right) + f_0^{in}(0)$$

Prandtl's matching condition requires that

$$\lim_{s \to +\infty} f_0^{in}(s) = \lim_{x \to 0_+} f_0^{outer,+}(x) = K^+ = 2$$

and $$\lim_{s \to -\infty} f_0^{in}(s) = \lim_{x \to 0-} f_0^{outer,-}(x) = K^- = 1$$

Since $\lim_{x \to +\infty} \operatorname{erf}(x) = 1$ and $\lim_{x \to -\infty} \operatorname{erf}(x) = -1$,

this implies that

$$\begin{cases} K_{in}\sqrt{\frac{\pi}{2}} + f_0^{in}(0) = 2 \\ -K_{in}\sqrt{\frac{\pi}{2}} + f_0^{in}(0) = 1 \end{cases} \Rightarrow \begin{cases} f_0^{in}(0) = \frac{3}{2} \\ K_{in} = \frac{1}{2}\sqrt{\frac{2}{\pi}} = \frac{1}{\sqrt{2\pi}} \end{cases}$$

so finally, $f_0^{in}(s) = \frac{3}{2} + \frac{1}{2}\,\mathrm{erf}\left(\frac{s}{\sqrt{2}}\right)$

Because we now have 3 components to the solution instead of just two, the formula for the composite expansion is a bit different but based on the same idea:

$$f_{composite}(x) = \begin{cases} f_+^{outer}(x) + f^{in}(x) - \lim\limits_{s\to+\infty} f^{in}(s) & \text{if } x>0 \\ f_-^{outer}(x) + f^{in}(x) - \lim\limits_{s\to-\infty} f^{in}(s) & \text{if } x<0 \end{cases}$$

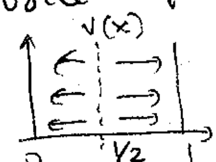
$$= \begin{cases} 2e^{-x} + \frac{3}{2} + \frac{1}{2}\,\mathrm{erf}\left(\frac{x}{\sqrt{2\varepsilon}}\right) - 2 = 2e^{-x} - \frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{x}{\sqrt{2\varepsilon}}\right) \\ e^{-x} + \frac{3}{2} + \frac{1}{2}\,\mathrm{erf}\left(\frac{x}{\sqrt{2\varepsilon}}\right) - 1 = e^{-x} + \frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{x}{\sqrt{2\varepsilon}}\right) \end{cases}$$

which is not very pretty. However, we also see that

$$f(x)_{composite} = \left(\frac{3}{2} + \frac{1}{2}\,\mathrm{erf}\left(\frac{x}{\sqrt{2\varepsilon}}\right)\right)e^{-x}$$

has all the right behavior. → This is an equivalent, but more compact composite expansion

<u>Example 4</u>: Boundary layers on each side:




This example illustrates that things are not necessarily so straightforward.

Consider
$$\begin{cases} \varepsilon f'' - 2\left(x - \frac{1}{2}\right) f' = 0 \\ f(0) = a \quad f(1) = b \end{cases}$$

- The outer expansion is of the form $f_0 + \varepsilon f_1 + \cdots$

  So $-2\left(x - \frac{1}{2}\right) f_0' = 0 \Rightarrow f_0(x) = \text{constant}$

  However, we cannot match $f_0$ to the boundary conditions since each of them is within its respective boundary layer.

→ Here we say that the outer expansion is "<u>detached</u>" from the boundaries.

→ Let's hope that we can find the constant by matching with the two inner expansions.

Now let's deal with the inner expansion near $x=0$: Call it $f^{(0)}(s)$

let $s = \frac{x}{\varepsilon^p}$ $\Rightarrow$

$$\varepsilon^{1-2p}\frac{d^2 f^{(0)}}{ds^2} - 2\left(\varepsilon^p s - \frac{1}{2}\right)\varepsilon^{-p}\frac{df^{(0)}}{ds} = 0$$

$\varepsilon^p s$ is always $\ll$

Principle of least degeneracy: we require that

$$\varepsilon^{1-2p} = \varepsilon^{-p} \Rightarrow \quad 1-2p = -p \Rightarrow \quad p=1$$

Then, to lowest order

$$\frac{d^2 f_0^{(0)}}{ds^2} + \frac{df_0^{(0)}}{ds} = 0 \quad \Longrightarrow \quad f_0^{(0)}(s) = A^{(0)}e^{-s} + B^{(0)}$$

To satisfy the BC @ $x=0$ $(s=0)$ we take $A^{(0)} + B^{(0)} = a$

$\Rightarrow$ $f_0^{(0)}(s) = A^{(0)}(e^{-s} - 1) + a$ $\qquad$ $A^{(0)}$ remains to be determined.

Finally, let's deal with the inner expansion near $x=1$; call it $f^{(1)}(\eta)$

let $\eta = \frac{1-x}{\varepsilon^p}$ $\rightarrow$ $\frac{d}{dx} = \frac{d\eta}{dx}\frac{d}{d\eta} = -\frac{1}{\varepsilon^p}\frac{d}{d\eta}$

$$\Rightarrow \quad \varepsilon^{1-2p}\frac{d^2 f^{(1)}}{d\eta^2} + 2\left(1 - \varepsilon^p\eta - \frac{1}{2}\right)\varepsilon^{-p}\frac{df^{(1)}}{d\eta} = 0$$

$\Rightarrow$ For the same reasons as above, we need

$\varepsilon^{1-2p} = \varepsilon^{-p}$ $\Rightarrow$ $p=1$ so, to lowest order

$$\frac{d^2 f_0^{(1)}}{d\eta^2} - \frac{df_0^{(1)}}{d\eta} = 0 \quad \Rightarrow \quad f_0^{(1)}(\eta) = A^{(1)}e^{s} + B^{(1)}$$

To satisfy the BCs, we need

$$f_0^{(1)}(0) = A^{(1)} + B^{(1)} = b \Rightarrow \quad B^{(1)} = b - A^{(1)}$$

so $\quad f_0^{(1)}(\eta) = A^{(1)}(e^{s} - 1) + b$

(recall that $\eta = 0$ when $x = 1$)

So far, everything look fine and it seems we just have to match them all to one another.

However, even without doing the calculation we now see that there is a problem: we will be getting 2 matching conditions, but we have 3 variables to fit! (one from the outer expansion, and one for each inner expansion).

$\rightarrow$ this problem has an $\infty$ of possible solutions!!

The issue with this problem is that it is almost ill-posed. This is not a problem with the expansion, but rather, a problem with the original equation.

Not all problems with 2 boundary layers suffer from this issue, however. Let's look at a final example that is well-posed.

Example 5

$$\begin{cases} \varepsilon f'' - f = -2\sin(x-\frac{1}{2}) \\ f(0) = 0 \qquad f(1) = 0. \end{cases}$$

In this example, the "steady-state of a time-dependent advection/diffusion problem" analogy does not help us find the location of the bl, since there is no $\frac{df}{dx}$ term in the equation.

However, note that when $\varepsilon = 0$ $f = 2\sin(x-\frac{1}{2})$ which does not satisfy either of the boundary conditions $\Rightarrow$ the system will therefore have to have one boundary layers on each side to fit the outer expansion to the b.cs.

Outer expansion: if $f = f_0 + \varepsilon f_1 + \cdots$

we have, to lowest order $f_0^{outer}(x) = 2\sin\left(x - \frac{1}{2}\right)$

→ this time, no arbitrary constants!

Inner expansion near $x=0$:

let $s = \frac{x}{\varepsilon^p}$ then (using the same notation as earlier)

$$\varepsilon^{1-2p}\frac{d^2 f^{(0)}}{ds^2} - f^{(0)} = -2\sin\left(\varepsilon^p s - \frac{1}{2}\right) \qquad \varepsilon^p s \ll \frac{1}{2} \text{ always}$$

→ Van Dykes matching condition requires $\varepsilon^{1-2p} = \varepsilon^0$

$\Rightarrow \quad 1-2p = 0 \Rightarrow \quad p = \frac{1}{2}$

To lowest order, we then get

$$\frac{d^2 f_0^{(0)}}{ds^2} - f_0^{(0)} = 2\sin\left(\tfrac{1}{2}\right)$$

so $\quad f_0^{(0)}(s) = A^{(0)} e^{s} + B^{(0)} e^{-s} - 2\sin\left(\tfrac{1}{2}\right)$

To fit the BC at $s=0$ $(x=0)$ we have

$$A^{(0)} + B^{(0)} - 2\sin\left(\tfrac{1}{2}\right) = 0 \Rightarrow$$

$$B^{(0)} = -A^{(0)} + 2\sin\left(\tfrac{1}{2}\right)$$

$$\Rightarrow \quad f^{(0)}(s) = A^{(0)}\left(e^{s} - e^{-s}\right) - 2\sin\left(\tfrac{1}{2}\right)\left(1 - e^{-s}\right)$$

Inner expansion near $x=1$

let $\eta = \frac{1-x}{\varepsilon^p}$

$$\Rightarrow \quad \varepsilon^{1-2p}\frac{d^2 f^{(1)}}{d\eta^2} - f^{(1)} = -2\sin\left(1 - \eta\varepsilon^p - \frac{1}{2}\right)$$

Again this requires $p = \frac{1}{2} \Rightarrow$ to lowest order

$$\frac{d^2 f_0^{(1)}}{d\eta^2} - f_0^{(1)} = -2\sin\left(\tfrac{1}{2}\right)$$

The solution is $f_0^{(1)}(\eta) = A^{(1)}e^{s} + B^{(1)}e^{-s} + 2\sin(\frac{1}{2})$

At $x=1$ $(\eta=0)$, $f_0^{(1)}(0)=0 \Rightarrow A^{(1)} + B^{(1)} + 2\sin(\frac{1}{2}) = 0$

so $B^{(1)} = -A^{(1)} - 2\sin(\frac{1}{2}) \Rightarrow$

$$f_0^{(1)}(\eta) = A^{(1)}(e^{\eta} - e^{-\eta}) + 2\sin(\tfrac{1}{2})(1-e^{-\eta})$$

This time, the two matching conditions $x=0$ and $x=1$ will indeed constrain the two unknown constants $A^{(0)}$ and $A^{(1)}$. Let's proceed with Prandtl's matching condition near $x=0$ first. We want

$$\lim_{x\to 0} f^{outer}(x) = \lim_{s\to +\infty} f^{(0)}(s)$$

$$\Rightarrow \lim_{x\to 0} 2\sin(x-\tfrac{1}{2}) = \lim_{s\to +\infty} A^{(0)}(e^{s}-e^{-s}) - 2\sin(\tfrac{1}{2})(1-e^{-s})$$

$$-2\sin(\tfrac{1}{2}) = \lim_{s\to +\infty} A^{(0)}e^{s} - 2\sin(\tfrac{1}{2})$$

The only way this can work is if $A^{(0)}=0$

$$\Rightarrow f_0^{(0)}(s) = -2\sin(\tfrac{1}{2})(1-e^{-s}) \rightarrow \underset{x\to 0}{f_0^{(0)}(x)} = -2\sin(\tfrac{1}{2})(1-e^{-\frac{x}{\sqrt{\varepsilon}}})$$

Near $x=1$, we want

$$\lim_{x\to 1} f^{outer}(x) = \lim_{\eta\to +\infty} f^{(1)}(\eta)$$

$$\Rightarrow \lim_{x\to 1} 2\sin(x-\tfrac{1}{2}) = \lim_{\eta\to +\infty} A^{(1)}(e^{\eta}-e^{-\eta}) + 2\sin(\tfrac{1}{2})(1-e^{-\eta})$$

$$2\sin(\tfrac{1}{2}) = \lim_{\eta\to +\infty} A^{(1)}e^{\eta} + 2\sin(\tfrac{1}{2})$$

$\Rightarrow$ again this will only work if $A^{(1)}=0$

$$\rightarrow f_0^{(1)}(\eta) = 2\sin(\tfrac{1}{2})(1-e^{-\eta}) \rightarrow \underset{x\to 1}{f_0^{(1)}(x)} = 2\sin(\tfrac{1}{2})(1-e^{\frac{x-1}{\sqrt{\varepsilon}}})$$

$\Rightarrow$ We see that in this case, there is a solution even though the outer expansion is detached from the boundaries. That's because the outer expansion itself has no arbitrary constants.

Again, because we have 3 parts to match, the composite expansion formula is a little different

$$f_{composite}(x) = f_{outer}(x) + f^{(0)}(x) - \lim_{s \to +\infty} f^{(0)}(s) + f^{(1)}(x) - \lim_{\eta \to +\infty} f^{(1)}(\eta)$$

$$= 2\sin\left(x - \tfrac{1}{2}\right) + 2\sin\left(\tfrac{1}{2}\right)\left(e^{-\frac{x}{\sqrt{\varepsilon}}} - 1\right) + \cancel{2\sin\left(\tfrac{1}{2}\right)} + 2\sin\left(\tfrac{1}{2}\right)\left(1 - e^{\frac{x-1}{\sqrt{\varepsilon}}}\right) - \cancel{2\sin\left(\tfrac{1}{2}\right)}$$

$$= 2\sin\left(x - \tfrac{1}{2}\right) + 2\sin\left(\tfrac{1}{2}\right)\left(e^{-\frac{x}{\sqrt{\varepsilon}}} - e^{\frac{x-1}{\sqrt{\varepsilon}}}\right)$$

## V <u>Higher-order matching: Van Dyke's matching principle</u>

So far we have only been concerned with obtaining $0^{th}$ order composite expansions (1-term expansion). We saw that Prandtl's matching condition does not work "as is" if we want to keep more terms, however.

A nice technique that works most of the time for higher-order matched asymptotic expansions is Van-Dyke's matching principle.

<u>Idea</u>:
- <u>Taylor expand</u> the outer expansion as $x \to x_{BL}$ (where $x_{BL}$ = position of BL)
- substitute $x$ with $x_{BL} + s\varepsilon^{p}$

$\left( \text{if } s = \frac{x - x_{BL}}{\varepsilon^{p}} \to x = x_{BL} + s\varepsilon^{p} \right)$

This procedure gives the "inner limit of the outer expansion".

- Then compare this with the expansion of the inner function as $s \to \pm\infty$ (depending on the structure of the BL)
  $\to$ this gives "the outer limit of the inner expansion".

- Match term by term, and voila!

Example 1. Let's go back to
$$\begin{cases} \varepsilon \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = 0 \\ y(0) = 0 \qquad y(1) = 1 \end{cases}$$

We had:
$$\begin{cases} y^{outer}(x) = e^{1-x} + \varepsilon(1-x)e^{1-x} \quad \text{(2 term)} \\ y^{inner}(s) = A_0(1-e^{-s}) + \varepsilon A_1(e^{-s}-1) \\ \qquad - \varepsilon A_0 s(1+e^{-s}) \quad \text{(2 term)} \end{cases}$$

① Taylor-expand $y^{outer}(x)$ as $x \to 0$:
$$y^{outer}(x) = e \cdot e^{-x} + \varepsilon(1-x)e\, e^{-x}$$
$$= e\left(1 - x + \frac{x^2}{2} + \cdots\right) + \varepsilon(1-x)e\left(1 - x + \frac{x^2}{2} + \cdots\right)$$

② Substitute $x = \varepsilon s$
$$y^{outer}(s) = e\left(1 - \varepsilon s + \frac{\varepsilon^2 s^2}{2} \cdots\right) + \varepsilon(1-\varepsilon s)e\left(1 - \varepsilon s + \frac{\varepsilon^2 s^2}{2} \cdots\right)$$
$$= e + \varepsilon\left[-es + e\right] + O(\varepsilon^2)$$

③ Take the taylor expansion of $y^{inner}(s)$ as $s \to +\infty$

Note that as $s \to +\infty$, all terms in $e^{-s}$ are asymptotically small so we simply have
$$y^{inner}(s) = A_0 + \varepsilon A_1(-1) - \varepsilon A_0 s + O(\varepsilon^2)$$
$$= A_0 + \varepsilon\left[-A_1 - A_0 s\right]$$

(4) Match order by order.

$$\begin{cases} A_0 = e & \to \text{as in Prandtl's matching condition} \\ -A_1 - A_0 s = e(1-s) \end{cases}$$

$$\Rightarrow \begin{cases} A_0 = e \\ A_1 = -es - e(1-s) = -e \end{cases}$$

Note how the terms in s cancel out. This is crucial, but not always guaranteed. If this does not happen, then either we made an error in the algebra (alwas a possibility) or it is a symptom of the failure of Van Dyke's matching method.

See Hinch textbook for examples.

Finally, we can construct the composite expansion as

$$y(x) = \underbrace{y^{outer}(x)}_{2-\text{term}} + \underbrace{y^{inner}(x)}_{2-\text{term}} - \underbrace{L}_{2-\text{term}}$$

← their common limit

$$= e^{1-x} + \varepsilon(1-x)e^{1-x} + e(1+\varepsilon)(1-e^{-\frac{x}{\varepsilon}})$$

$$- \varepsilon e\frac{x}{\varepsilon}(1+e^{-\frac{x}{\varepsilon}}) - e - \varepsilon e(1-\frac{x}{\varepsilon})$$

$$= e^{1-x} - e^{1-\frac{x}{\varepsilon}} - exe^{-\frac{x}{\varepsilon}}$$

$$+ \varepsilon\left[(1-x)e^{1-x} - e^{1-\frac{x}{\varepsilon}}\right] + \cdots$$

Note the appearance of an extra term in the 0th-order composite expansion → latter was not uniform.

Example 2 A nonlinear problem : same procedure ...

Let $\begin{cases} \varepsilon y'' + y' + y^2 = 0 \\ y(0) = 1 \quad y(1) = 0 \end{cases}$ $\quad \varepsilon < 0$

- first let's use the new small parameter $\eta = -\varepsilon$ then

$$\begin{cases} \eta y'' - y' - y^2 = 0 \\ y(0) = 1 \quad y(1) = 0 \end{cases} \qquad \varepsilon > 0$$

this then shows that the advection velocity is $> 0$, so the boundary layer is to the right. (at $x = 1$).

- Outer expansion:

let $y = y_0 + \eta y_1 + \cdots$

$\Rightarrow$ 0th: $\begin{cases} y_0' + y_0^2 = 0 \\ y_0(0) = 1 \end{cases}$ $\leftarrow$ we only apply the one that is not in the BL.

$$\Rightarrow -\frac{dy_0}{y_0^2} = dx \Rightarrow \frac{1}{y_0} = x + K \Rightarrow y_0 = \frac{1}{K + x}$$

to apply the BC we need $1 = \frac{1}{K}$ so $\boxed{y_0 = \frac{1}{1+x}}$

$\Rightarrow$ 1st: $\begin{cases} y_0'' - y_1' - 2y_0 y_1 = 0 \\ y_1(0) = 0 \end{cases}$

$$\Rightarrow \quad y_1' + 2\frac{y_1}{1+x} = \left[\frac{-1}{(1+x)^2}\right]' = \frac{2}{(1+x)^3}$$

Let's use an integrating factor:

$$\mu(x) = e^{\int \frac{2}{1+x} dx} = e^{2\ln(1+x)} = (1+x)^2$$

$$\Rightarrow \quad \frac{d}{dx}\left((1+x)^2 y_1\right) = \frac{2}{1+x}$$

So $(1+x)^2 y_1 = 2\ln(1+x) + K$

to apply the BC, $y_1(0) = 0 \Rightarrow K = 0$ so

$$y_1(x) = \frac{2\ln(1+x)}{(1+x)^2}$$

So the outer expansion, to this order, is

$$y^{outer}(x) = \frac{1}{1+x} + 2\eta \frac{\ln(1+x)}{(1+x)^2} + o(\varepsilon^2)$$

- Inner expansion. The BL is at $x=1$ so we let $s = \frac{x-1}{\eta^p}$ $\Rightarrow$ let's use Van Dyke's principle to

find p: $\eta^{1-2p} \frac{d^2y}{ds^2} - \eta^{-p} \frac{dy}{ds} - y^2 = 0$

For $p>0$, (and $y$ of order 1), $\eta^{-p}$ is alway much larger than 1 So, to the lowest order, we should have a balance between

$$\eta^{1-2p} = \eta^{-p} \Rightarrow 1-2p = -p$$
$$\Rightarrow 1-p = 0 \Rightarrow p = 1$$

So let $s = \frac{x-1}{\eta}$

Also let $y^{inner} = y_0^{in} + \eta y_1^{in} + \dots$

To lowest order: $\frac{1}{\varepsilon} \frac{d^2 y_0^{in}}{ds^2} - \frac{1}{\varepsilon} \frac{dy_0^{in}}{ds} = 0$ , with $y_0^{in}(0) = 0$

$$\Rightarrow \quad y_0^{in}(s) = A_0 e^s + B_0$$

to match this to the boundary condition at $x=1$ $(s=0)$ we need $A_0 + B_0 = 0 \Rightarrow$

$$y_0^{in}(s) = A_0(e^s - 1).$$

To the next order:

$$\frac{d^2 y_1^{in}}{ds^2} - \frac{dy_1^{in}}{ds} - (y_0^{in})^2 = 0 \qquad \text{with } y_1^{in}(0) = 0$$

$$\Rightarrow \quad \frac{d^2 y_1^{in}}{ds^2} - \frac{dy_1^{in}}{ds} = A_0^2 (e^s - 1)^2 = A_0^2 (e^{2s} - 2e^s + 1)$$

The general solution to the homogeneous problem is

$$y_1^{in}(s) = A_1 e^s + B_1$$

The particular solution will be of the form

$$y_{PS} = sK_1 + sK_2 e^s + K_3 e^{2s} + K_4$$

$K_4$: fold into $B_1$

$$\frac{dy_{PS}}{ds} = K_1 + K_2 e^s + sK_2 e^s + 2K_3 e^{2s}$$

$$\frac{d^2 y_{PS}}{ds^2} = 2K_2 e^s + sK_2 e^s + 4K_3 e^{2s}$$

$$\Rightarrow \quad 2K_2 e^s + \cancel{sK_2 e^s} + 4K_3 e^{2s} - K_1 - K_2 e^s - \cancel{sK_2 e^s} - 2K_3 e^{2s}$$
$$= A_0^2 (e^{2s} - 2e^s + 1)$$

$$\Rightarrow \quad 2K_3 = A_0^2, \quad K_2 = -2A_0^2, \quad -K_1 = +A_0^2$$

and so finally,

$$y_1^{in}(s) = A_1 e^s + B_1 \quad + A_0^2\left(-s - 2se^s + \tfrac{1}{2}e^{2s}\right)$$

To fit the BC, we need: $A_1 + B_1 + \frac{1}{2}A_0^2 = 0$

so $B_1 = -\frac{1}{2}A_0^2 - A_1$

$$\Rightarrow y_1^{in}(s) = A_1(e^s - 1) + A_0^2\left(-s - 2se^s + \tfrac{1}{2}e^{2s} - \tfrac{1}{2}\right)$$
$$= A_1(e^s - 1) + A_0^2\left(\tfrac{1}{2}(e^{2s} - 1) - s(1 + 2e^s)\right)$$

And finally: $y^{inner}(s) = A_0(e^s - 1) + \eta A_1(e^s - 1) + \eta A_0^2\left[\tfrac{1}{2}(e^{2s} - 1) - s(1 + 2e^s)\right]$

36. Let's now take the Taylor expansion of $y^{inner}(s)$ as $s \to -\infty$ (indeed, $x \to$ out of BL corresponds to $s \to -\infty$)

$$\Rightarrow \lim_{s\to-\infty} y^{inner}(s) = -A_0 - \eta A_1 \quad + \eta A_0^2\left(-\frac{1}{2} - s\right) + o(\eta^2)$$

And the Taylor expansion of $y^{outer}$ as $x \to 1$:

$$y^{outer}(x) \simeq y^{outer}(1) + \quad (x-1)\frac{dy^{outer}}{dx}\Big|_{x=1}$$

$$= \frac{1}{2} + \eta\frac{2\ln 2}{(2)^2} + (x-1)\left[-\frac{1}{(2)^2} + \frac{2\eta}{(2)^3} + 2\ln 2\,\eta\left(-\frac{2}{(2)^3}\right.\right.$$

$$= \frac{1}{2} + \eta\frac{\ln 2}{2} + (x-1)\left[-\frac{1}{4} + \frac{\eta}{4} - \frac{\ln 2}{2}\eta\right] + o(\eta^2)$$

Matching the two expansions, using $x-1 = \eta s$, yields:

$$\lim y^{outer}(s) = \frac{1}{2} + \frac{\ln 2}{2}\eta - \frac{\eta s}{4} + o(\eta^2)$$

$$= \lim y^{inner}(s) = -A_0 - \eta A_1 + \eta A_0^2\left(-\frac{1}{2} - s\right) + o(\eta^2)$$

At order $\eta^0$, we get $\boxed{A_0 = -\frac{1}{2}}$

At order $\eta^1$, we get

$$+\frac{\ln 2}{2} - \frac{s}{4} = -A_1 - \left(\frac{1}{2} + s\right)A_0^2$$

$$= -A_1 - \left(\frac{1}{2} + s\right)\frac{1}{4}$$

$$\Rightarrow \frac{\ln 2}{2} = -A_1 - \frac{1}{8} \Rightarrow \boxed{A_1 = -\frac{\ln 2}{2} - \frac{1}{8}}$$

So $$y^{inner}(s) = -\frac{1}{2}(e^s - 1) - \eta\left(\frac{\ln 2}{2} + \frac{1}{8}\right)(e^s - 1) + \frac{\eta}{4}\left[\frac{1}{2}(e^{2s} - 1) - s(1 + e^{s})\right]$$

And finally, the composite 2-term expansion is:

$$y_{composite}(x) = \frac{1}{1+x} + 2\eta\frac{\ln(1+x)}{(1+x)^2} - \frac{1}{2}\left(e^{\frac{x-1}{\eta}} - 1\right) - \eta\left(\frac{\ln 2}{2} + \frac{1}{8}\right)\left(e^{\frac{x-1}{\eta}} - 1\right) +$$

$$\frac{\eta}{4}\left[\frac{1}{2}\left(e^{\frac{2(x-1)}{\eta}}-1\right)-\frac{x-1}{\eta}\left(1+2e^{\frac{x-1}{\eta}}\right)\right] \qquad +O(\eta^2)$$

$$-\left(\frac{1}{2}+\eta\frac{\ln 2}{2}-\frac{\eta}{4}\frac{x-1}{\eta}\right) \text{ — 2-term}$$

common limit of the two expansions.

$$= \frac{1}{1+x}-\frac{1}{2}\left(e^{\frac{x-1}{\eta}}-1\right)-\frac{x-1}{4}\left(1+2e^{\frac{x-1}{\eta}}\right)-\frac{1}{2}+\frac{1}{4}(x-1)$$
$$+\eta\left[2\frac{\ln(1+x)}{(1+x)^2}-\left(\frac{\ln 2}{2}+\frac{1}{8}\right)\left(e^{\frac{x-1}{\eta}}-1\right)+\frac{1}{8}\left(e^{\frac{2(x-1)}{\eta}}-1\right)-\frac{\ln 2}{2}\right.$$
$$+O(\eta^2)$$

$$= \frac{1}{1+x}-\frac{1}{2}e^{\frac{x-1}{\eta}}-\frac{x-1}{2}e^{\frac{x-1}{\eta}}$$
$$+\eta\left[2\frac{\ln(1+x)}{(1+x)^2}-\left(\frac{\ln 2}{2}\right)e^{\frac{x-1}{\eta}}+\frac{1}{8}\left(e^{\frac{2(x-1)}{\eta}}-e^{\frac{x-1}{\eta}}\right)\right]$$

$$= \frac{1}{1+x}-\frac{x}{2}e^{\frac{x-1}{\eta}}$$
$$+\eta\left[2\frac{\ln(1+x)}{(1+x)^2}-\frac{\ln 2}{2}e^{\frac{x-1}{\eta}}+\frac{1}{8}\left(e^{\frac{2(x-1)}{\eta}}-e^{\frac{x-1}{\eta}}\right)\right].$$