HOMEWORK 3

Problem 1 :

$$\frac{d^2u}{dt^2} + u = \varepsilon u\left(\frac{du}{dt}\right)^2 \qquad u(0)=1 \qquad \frac{du}{dt}(0)=0$$

let $\quad u = u_0 + \varepsilon u_1 + \cdots$

$\quad \tau = t(1+ a_1\varepsilon + a_2\varepsilon^2 + \cdots)$

$$\Rightarrow \quad (1+a_1\varepsilon + a_2\varepsilon^2+\cdots)^2 \frac{d^2}{d\tau^2}(u_0 + \varepsilon u_1 + \cdots) + (u_0+\varepsilon u_1+\cdots)$$

$$= \varepsilon(u_0+\varepsilon u_1+\cdots)\underbrace{\left[(1+a_1\varepsilon + a_2\varepsilon^2+\cdots)\frac{d}{d\tau}(u_0+\varepsilon u_1+\cdots)\right]^2}_{\underbrace{\left[\frac{du_0}{d\tau} + \varepsilon\left(a_1\frac{du_0}{d\tau} + \frac{du_1}{d\tau}\right)+\cdots\right]^2}_{\left(\frac{du_0}{d\tau}\right)^2 + 2\varepsilon\left(\frac{du_0}{d\tau}\right)\left(a_1\frac{du_0}{d\tau}+\frac{du_1}{d\tau}\right)+\cdots}}$$

To zeroth order:

$$\begin{cases} \dfrac{d^2u_0}{d\tau^2} + u_0 = 0 \\ u_0(0) = 1 \\ \dfrac{du_0}{d\tau} = 0 \end{cases}$$

$\rightarrow u_0 = \cos\tau$ as usual

To first order

$$2a_1\frac{d^2u_0}{d\tau^2} + \frac{d^2u_1}{d\tau^2} + u_1 = u_0\left(\frac{du_0}{d\tau}\right)^2$$

$$\Rightarrow \quad \frac{d^2u_1}{d\tau^2} + u_1 = \cos\tau\sin^2\tau + 2a_1\cos\tau$$

$$= \frac{1}{4}\cos\tau - \frac{1}{4}\cos 3\tau + 2a_1\cos\tau$$

$\Rightarrow$ to elemenate the source of the secular term, we pick $\frac{1}{4} + 2a_1 = 0 \Rightarrow a_1 = -\frac{1}{8}$

Then $\dfrac{d^2u_1}{d\tau^2} + u_1 = -\dfrac{1}{4}\cos 3\tau$ with ic $\begin{cases}\dfrac{du_1}{d\tau} = 0 \\ u_1 = 0\end{cases}$ at $\tau=0$

$$\rightarrow \quad u_1(\tau) = A\cos\tau + B\sin\tau + K\cos 3\tau$$

where K satisfies

$$-9K + K = -\frac{1}{4} \Rightarrow -8K = -\frac{1}{4} \Rightarrow K = \frac{1}{32}$$

A & B satisfy: $\quad A + K = 0$
$\quad B = 0$

$$\text{so} \quad u_1(\tau) = \frac{1}{32}\left(\cos 3\tau - \cos\tau\right) \Rightarrow \boxed{u = \cos\tau + \frac{\varepsilon}{32}\left(\cos 3\tau - \cos\tau\right)}$$

Finally, we need to go to next order to make this a <u>uniform</u> 2-term expansion:

<u>to second order</u>

$$\begin{cases} (a_1^2 + 2a_2)\dfrac{d^2u_0}{d\tau^2} + 2a_1\dfrac{d^2u_1}{d\tau^2} + \dfrac{d^2u_2}{d\tau^2} + u_2 \\ \quad = u_1\left(\dfrac{du_0}{d\tau}\right)^2 + 2u_0\dfrac{du_0}{d\tau}\left(a_1\dfrac{du_0}{d\tau} + \dfrac{du_1}{d\tau}\right) \\ \dfrac{du_2}{d\tau} = u_2 = 0 \text{ at } \tau = 0 \end{cases}$$

$$\Rightarrow \frac{d^2u_2}{d\tau^2} + u_2 = +(a_1^2 + 2a_2)\cos\tau + \frac{2a_1}{32}\left(9\cos 3\tau - \cos\tau\right)$$

$$+ \frac{1}{32}\left(\cos 3\tau - \cos\tau\right)\sin^2\tau + 2\cos\tau(-\sin\tau)\cdot$$

$$\cdot\left(-a_1\sin\tau - \frac{1}{32}\left(3\sin 3\tau - \sin\tau\right)\right)$$

$$= (a_1^2 + 2a_2)\cos\tau + \frac{2a_1}{32}\left(9\cos 3\tau - \cos\tau\right)$$

$$+ \frac{1}{32}\left[-\frac{1}{4}\cos\tau + \frac{1}{2}\cos 3\tau - \frac{1}{4}\cos 5\tau\right] + \left(2a_1 - \frac{1}{32}\right)\left(\frac{1}{4}\cos\tau - \frac{1}{4}\cos 3\tau\right)$$

$$+ \frac{2}{32}\left(\frac{3}{4}\cos\tau - \frac{3}{4}\cos 5\tau - \frac{1}{4}\cos\tau + \frac{1}{4}\cos 3\tau\right)$$

$\Rightarrow$ to eliminate all sources of secular terms, we need

$$a_1^2 + 2a_2 - \cancel{\frac{2a_1}{32}} - \cancel{\frac{1}{4\times 32}} + \left(2a_1 - \frac{1}{32}\right)\frac{1}{4} + \frac{1}{32}$$

. which simplifies to

$$2a_2 + \frac{1}{64} + \left(-\frac{1}{4} - \frac{1}{32}\right)\cdot\frac{1}{4} + \frac{1}{32}$$

$$2a_2 = \frac{3}{128} \Rightarrow a_2 = \frac{3}{256} \Rightarrow \boxed{\tau = \left(1 - \frac{\varepsilon}{8} + \frac{3\varepsilon^2}{256} + \cdots\right)}$$

Problem 2: See textbook page 85 for solution.

Problem 3 $\quad \dfrac{d^2u}{dt^2} + u = \varepsilon u\left(\dfrac{du}{dt}\right)^4 \qquad u(0) = 0 \quad \dfrac{du}{dt}(0) = 1$

First get 2-term standard expansion: let

$u = u_0 + \varepsilon u_1$

→ to zeroth order:

$$\begin{cases} \dfrac{d^2u_0}{dt^2} + u_0 = 0 \\ u_0(0) = 0 \\ \dfrac{du_0}{dt}(0) = 1 \end{cases} \rightarrow u_0(t) = \sin t$$

→ to first order

$$\frac{d^2u_1}{dt^2} + u_1 = u_0\left(\frac{du_0}{dt}\right)^4 \qquad \text{with IC } \begin{cases} u_1(0) = 0 \\ \dfrac{du_1}{dt}(0) = 0 \end{cases}$$

$$= \sin t \cdot (\cos t)^4$$

$$= \frac{2}{16}\sin t + \frac{3}{16}\sin 3t + \frac{1}{16}\sin 5t$$

So $\quad u_1(t) = A\sin t + B\sin t + Ct\cos t + D\sin 3t + E\sin 5t$

where: C satisfies:

$$-2C\sin t - \cancel{Ct\cos t} + \cancel{Ct\cos t} = \frac{2}{16}\sin t$$

$$\rightarrow C = -\frac{1}{16}$$

D satisfies:

$$-9D + D = \frac{3}{16} \quad \text{so} \quad D = -\frac{3}{8\times16} = -\frac{3}{128}$$

E satisfies:

$$-25E + E = \frac{1}{16} \quad \text{so}$$

$$E = \frac{-1}{24\times 16}$$

so $u_1(t) = A\cos t + B\sin t - \frac{1}{16}\left(t\cos t + \frac{3}{8}\sin 3t + \frac{1}{24}\sin 5t\right)$

$u_1(0) = 0 \Rightarrow A = 0$

$\frac{du_1}{dt}(0) = 0 \Rightarrow B - \frac{1}{16}\left(1 + \frac{9}{8} + \frac{5}{24}\right) = 0$

$$\Rightarrow B = \frac{7}{48} = \frac{1}{16}\cdot\frac{7}{3}$$

so finally:

$$u(t) = \sin t + \frac{\epsilon}{16}\left\{\frac{7}{3}\sin t - t\cos t - \frac{3}{8}\sin 3t - \frac{1}{24}\sin 5t\right\}$$

We now let $t = \tau + \omega_1(\tau)\varepsilon + \cdots$

so

$$u(\tau) = \sin(\tau + \omega_1(\tau)\varepsilon + \cdots) + \frac{\epsilon}{16}\left\{\frac{7}{3}\sin\tau - \tau\cos\tau - \frac{3}{8}\sin 3\tau - \frac{1}{24}\sin 5\tau\right\} + \cdots$$

$$= \sin\tau + \varepsilon\omega_1\cos\tau + \frac{\varepsilon}{16}\left\{\frac{7}{3}\sin\tau - \tau\cos\tau - \frac{3}{8}\sin 3\tau - \frac{1}{24}\sin 5\tau\right\} + \cdots$$

→ to eleminate the secular term, we let

$$\omega_1 = \frac{\tau}{16} \quad \text{so}$$

$$\begin{cases} u(\tau) = \sin\tau + O(\varepsilon) \\ t = \tau + \frac{\tau\varepsilon}{16} + O(\varepsilon^2) \end{cases} \rightarrow u(t) = \sin\left(\frac{t}{1+\frac{\varepsilon}{16}}\right) + O(\varepsilon)$$

Problem 4

(1) $$\begin{cases} f'' + \left(2 - \dfrac{1}{\sqrt{1+\epsilon^2 f^2}}\right) f = 0 \\ f(0) = 1 \\ f'(0) = 0 \end{cases}$$

Let's use renormalization; let $f = f_0 + \varepsilon f_1 + \ldots$ then

$$f_0'' + \varepsilon f_1'' + \ldots \left(2 - \frac{1}{\sqrt{1+\epsilon^2(f_0^2 + 2f_0 f_1 \varepsilon + \ldots)}}\right)(f_0 + \varepsilon f_1 + \ldots) = 0$$

This term is

$$\left(1 + \epsilon^2 (f_0^2 + 2\varepsilon f_0 f_1 + \ldots)\right)^{-1/2} = 1 - \frac{1}{2}\epsilon^2 (f_0^2 + 2\varepsilon f_0 f_1 + \ldots) + o(\varepsilon^4)$$

So to zeroth order:

$$\begin{cases} f_0'' + f_0 = 0 \\ f_0(0) = 1 \\ f_0'(0) = 0 \end{cases} \rightarrow f_0(t) = \cos t$$

to first order:

$$\begin{cases} f_1'' + \epsilon f_1 = 0 \\ f_1(0) = 0 \\ f_1'(0) = 0 \end{cases} \rightarrow$$ This is not good because it means $f_1(t) \equiv 0$.
So we need to go to next order:

to second order:

$$\begin{cases} f_2'' + f_2 + \frac{1}{2} f_0^3 = 0 \\ f_2(0) = 0 \\ f_2'(0) = 0 \end{cases} \rightarrow f_2'' + f_2 = -\frac{1}{2}\cos^2 t$$

$$= -\frac{1}{2}\left(\frac{3}{4}\cos t + \frac{1}{4}\cos 3t\right)$$

So $f_2(t) = A\cos t + B\sin t + Kt\sin t + C\cos 3t$

where K satisfies : $2K\cos t = -\frac{3}{8}\cos t \Rightarrow K = -3/16$

C satisfies : $-9C + C = -1/8 \Rightarrow C = 1/64$

A satisfies : $A + C = 0 \Rightarrow A = -1/64$

B satisfies : $B = 0$

so $f_2(t) = -\frac{1}{64}\cos t + \frac{1}{64}\cos 3t - \frac{3}{16} t\sin t$

and so $f(t) = \cos t + \frac{\epsilon^2}{16}\left[\frac{1}{4}\cos 3t - \frac{1}{4}\cos t - 3t\sin t\right] + \cdots$

Now let $t = \tau + \epsilon^2 \omega_2(\tau) + \cdots$ ← choose ϵ^2 by analogy with ea

$\Rightarrow f(\tau) = \cos(\tau + \epsilon^2\omega_2(\tau)) + \frac{\epsilon^2}{16}\left[\frac{1}{4}\cos 3\tau - \frac{1}{4}\cos\tau - 3\tau\sin\tau\right] + \cdots$

To eliminate secular term: $-\epsilon^2\omega_2(\tau)\sin\tau - \frac{\epsilon^2}{16}3\tau\sin\tau = 0$

$\Rightarrow \omega_2(\tau) = -\frac{3\tau}{16}$ so $t = \tau + \epsilon^2\left(-\frac{3\tau}{16}\right)$

$\Rightarrow$ Finally, the uniform 1-term expansion is: $f(t) = \cos\left(\frac{t}{1 - \frac{3\epsilon^2}{16}}\right)$

(2) $\begin{cases} f'' + f + (e^{\epsilon f} - 1) = 0 \\ f(0) = 1 \\ f'(0) = 0 \end{cases}$

let $f = f_0 + \epsilon f_1 + \cdots$

$\Rightarrow f_0'' + \epsilon f_1'' + \cdots + f_0 + \epsilon f_1 \cdots + (1 + \epsilon(f_0 + \epsilon f_1 + \cdots) + \cdots - 1) = 0$

to zeroth order $\Rightarrow \begin{cases} f_0'' + f_0 = 0 \\ f_0(0) = 1 \\ f_0'(0) = 0 \end{cases}$ so $f_0(t) = \cos t$

to 1st order: $\begin{cases} f_1'' + f_1 + \epsilon f_0 = 0 \\ f_1(0) = 0 \\ f_1'(0) = 0 \end{cases}$

$\Rightarrow f_1'' + f_1 = -\cos t$ so

$f_1(t) = A\cos t + B\sin t + Kt\sin t$

where K satisfies: $2K\cos t - Kt\sin t + Kt\sin t = -\cos t$

$\Rightarrow K = -\frac{1}{2}$

A satisfies: $A = 0$

B satisfies: $B = 0$

So $f_1(t) = -\frac{1}{2} \cdot t\sin t$

In this case, let's use renormalization to elemenate secular term: let $t = \tau + \varepsilon\omega_1(\tau) + \cdots$

$$\Rightarrow \quad f(t) = \cos(\tau + \varepsilon\omega_1(\tau) + \cdots) - \frac{\varepsilon}{2}\tau\sin\tau \quad + \cdots$$

$$= \cos\tau - \varepsilon\omega_1\sin\tau + \cdots - \frac{\varepsilon}{2}\tau\sin\tau \quad \cdots$$

$\Rightarrow$ we need $-\varepsilon\omega_1\sin\tau - \frac{\varepsilon}{2}\tau\sin\tau = 0$

so $\omega_1 = -\frac{\tau}{2}$

$\Rightarrow$ finally, $\begin{cases} f(\tau) = \cos\tau \\ t = \tau - \frac{\varepsilon\tau}{2} \end{cases} \Rightarrow f(t) = \cos\left(\frac{t}{1-\frac{\varepsilon}{2}}\right).$

<u>Problem 5</u>

$$\begin{cases} \frac{d^2u}{dt^2} + u + \varepsilon\left(\frac{du}{dt}\right)^3 = 0 \\ u(0) = a \\ \frac{du}{dt}(0) = 0 \end{cases}$$

$\Rightarrow$ let $u = u_0 + \varepsilon u_1 + \cdots$

to zeroth order:

$$\begin{cases} \frac{d^2u_0}{dt^2} + u_0 = 0 \\ u_0(0) = a \\ \frac{du_0}{dt}(0) = 0 \end{cases} \Rightarrow u_0(t) = a\cos t$$

to first order:

$$\begin{cases} \frac{d^2u_1}{dt^2} + u_1 + \left(\frac{du_0}{dt}\right)^3 = 0 \\ u_1(0) = 0 \\ du_1/dt(0) = 0. \end{cases}$$

$$\Rightarrow \quad \frac{d^2u_1}{dt^2} + u_1 = + a^3 \sin^3 t$$

$$= \frac{a^3}{4}(3\sin t - \sin 3t)$$

So $\quad u_1(t) = A\cos t + B\sin t + Kt\cos t + D\sin 3t$

where $\quad K$ satisfies:

$$-2K\sin t - \cancel{Kt\cos t} + \cancel{Kt\cos t} = \frac{3a^3}{4}\sin t$$

$$\Rightarrow \quad K = -\frac{3a^3}{8}$$

D satisfies: $\quad -9D + D = -\frac{a^3}{4} \Rightarrow D = \frac{a^3}{32}$

A satisfies: $\quad A = 0$

B satisfies: $\quad B + K + 3D = 0 \Rightarrow B = \frac{3a^3}{8} - \frac{3a^3}{32}$

$$= \frac{9a^3}{32}$$

So finally: $\quad u_1(t) = \frac{9a^3}{32}\sin t - \frac{3a^3}{8}t\cos t + \frac{a^3}{32}\sin 3t$

and $\quad u(t) = a\cos t + \frac{\epsilon a^3}{32}\left(9\sin t - 12t\cos t + \sin 3t\right)$

We now let $t = \tau + \omega_1(\tau)\varepsilon$ then

$$u(\tau) = a\cos(\tau + \omega_1(\tau)\varepsilon) + \frac{\epsilon a^3}{32}(9\sin\tau - 12\tau\cos\tau + \sin 3\tau)$$

$$= a\cos\tau - \omega_1(\tau)a\varepsilon\sin\tau + \frac{\varepsilon a^3}{32}(9\sin\tau - 12\tau\cos\tau + \sin 3$$

→ to eliminate the secular term we would need:

$$-\omega_1(\tau)a\sin\tau - \frac{a^3}{32}12\tau\cos\tau = 0$$

$$\Rightarrow \quad \omega_1(\tau) = -a^2 \cdot \frac{12\tau}{32}\frac{\cos\tau}{\sin\tau}$$

This is singular @ $\tau = \pi, 2\pi, 3\pi \ldots$ → we cannot get a uniform expansion in t from the 1-term solution in τ.