SOLUTIONS FOR HW1

Problem 4 p 17 $\quad x^3 - 4x = 2\varepsilon$

In this example the iterative method doesn't work, so let's just assume an asymptotic series of the kind

$$x = a_0 + a_1\varepsilon + a_2\varepsilon^2 + a_3\varepsilon^3 + \cdots$$

Then: $\quad (a_0 + a_1\varepsilon + a_2\varepsilon^2 + \cdots)^3 - 4(a_0 + a_1\varepsilon + \cdots) = 2\varepsilon$

$\Rightarrow$ 0th-order

$$a_0^3 - 4a_0 = 0$$

$\hookrightarrow \quad a_0(a_0^2 - 4) = 0$

$\hookrightarrow \quad a_0 = 0,\ 2 \text{ or } -2$

$\Rightarrow$ 1st order

$$3a_0^2 a_1 - 4a_1 = 2$$

$\hookrightarrow \quad a_1 = \dfrac{2}{3a_0^2 - 4}$

$\hookrightarrow \quad a_1 = +\dfrac{1}{2} \text{ or } -\dfrac{1}{4}$

So for two of the roots we already have a two-term expansion:

$$x = 2 - \frac{\varepsilon}{4}$$

$$\text{or } x = -2 - \frac{\varepsilon}{4}.$$

For the last root, however, we have to go further

$a_0 = 0$ so we can write

$$x = \tfrac{1}{2}\varepsilon + a_2\varepsilon^2 + \cdots$$

$$\left(\tfrac{1}{2}\varepsilon + a_2\varepsilon^2 + \cdots\right)^3 - 4\left(\tfrac{1}{2}\varepsilon + a_2\varepsilon^2 + \cdots\right) = -2\varepsilon$$

$$\Rightarrow \varepsilon^{\cancel{3}\,2}\left(\tfrac{1}{2} + a_2\varepsilon + \cdots\right)^3 - 4\cancel{\varepsilon}\left(\tfrac{1}{2} + a_2\varepsilon + \cdots\right) = -2\cancel{\varepsilon}$$

$\Rightarrow$ To the next order we get $a_2 = 0$! so we have to go to a_3

$$\varepsilon^2\left(\tfrac{1}{2} + a_3\varepsilon^2 + \cdots\right)^3 - 4\left(\tfrac{1}{2} + a_3\varepsilon^2 + \cdots\right) = -2$$

$$\Rightarrow \quad \tfrac{1}{8}\varepsilon^2 - 4a_3\varepsilon^2 = 0 \quad \text{to } O(\varepsilon^2)$$

$$\text{so} \quad a_3 = \frac{1}{32}$$

$\rightarrow$ finally the last root has $x = \frac{1}{2}\varepsilon + \frac{1}{32}\varepsilon^3$ (as a 2-term expansion)

<u>Optional Problem 7 page 17</u> $\qquad \varepsilon x^3 + x^2 + 2x + 1 = 0$

Here we expect a singular root. Let's first look for the non-singular ones with an iterative method.

- $[x^{(0)}]^2 + 2x^{(0)} + 1 = 0 \Rightarrow (x^{(0)}+1)^2 = 0 \Rightarrow x^{(0)} = -1$ as a double-root

- $\varepsilon[x^{(0)}]^3 + [x^{(1)}]^2 + 2x^{(1)} + 1 = 0$

$$\Rightarrow (x^{(1)}+1)^2 = -\varepsilon[x^{(0)}]^3 = \varepsilon$$

$$\rightarrow x^{(1)} = -1 \pm \sqrt{\varepsilon}$$

↳ clearly we are looking at an expansion in $\sqrt{\varepsilon}$ not ε

so let's look for $x = a_0 + \varepsilon^{1/2} a_1 + \varepsilon a_2 + \dots$

$$\varepsilon(a_0 + \varepsilon^{1/2} a_1 + \varepsilon a_2 + \dots)^3 + (a_0 + \varepsilon^{1/2} a_1 + \varepsilon a_2)^2 + 2(a_0 + \varepsilon^{1/2} a_1 + \varepsilon a_2 + \dots) + 1 = 0$$

To 0th order: $a_0^2 + 2a_0 + 1 = 0 \Rightarrow a_0 = -1$

To 1st order $(o(\varepsilon^{1/2}))$: $2a_0 a_1 + 2a_1 = 0 \Rightarrow (2a_0 + 2)a_1 = 0 \Rightarrow$ Trivial, does not say anything about a_1

To 2nd order $(o(\varepsilon))$: $a_0^3 + a_1^2 + \cancel{2a_0a_2} + \cancel{2a_2} = 0$

$$a_1^2 = -a_0^3 \Rightarrow a_1 = \pm 1$$

So the 2-term expansion is indeed (as found by iterative method) $x = -1 \pm \sqrt{\varepsilon}$ for the regular roots.

For the singular root, we expect it to be very large so we want to match εx^3 with x^2 to the lowest order $\Rightarrow \varepsilon x^3 + x^2 = 0 \Rightarrow x = -\frac{1}{\varepsilon}$

So let's try an expansion of the kind

$$x = \frac{a_{-1}}{\varepsilon} + a_0 + a_1\varepsilon + \dots$$

$$\varepsilon\left(\frac{a_{-1}}{\varepsilon} + a_0 + a_1\varepsilon + \dots\right)^3 + \left(\frac{a_{-1}}{\varepsilon} + a_0 + a_1\varepsilon + \dots\right)^2 + 2\left(\frac{a_{-1}}{\varepsilon} + a_0 + a_1\varepsilon + \dots\right) + 1 = 0$$

to $o\left(\frac{1}{\varepsilon^2}\right) \Rightarrow$ $(a_{-1})^3 + a_{-1}^2 = 0 \Rightarrow a_{-1} = -1$ as expected

to $o\left(\frac{1}{\varepsilon}\right)$ $3a_{-1}^2 a_0 + 2a_{-1}a_0 + 2a_{-1} = 0$

$$\Rightarrow a_0 = \frac{-2a_{-1}}{3a_{-1}^2 + 2a_{-1}} = \frac{-2}{3a_{-1} + 2} = \frac{-2}{-3+2}$$

$$= 2$$

$\Rightarrow$ Two term expansion is $x = -\frac{1}{\varepsilon} + 2$

Problem 6 p 17 $\quad \varepsilon x^3 + x^2 + 3x + 2 = 0$

There will be 1 singular root and 2 regular ones.

Let's look @ iterative method for regular root:

$$[x^{(0)}]^2 + 3x^{(0)} + 2 = 0$$

$$\rightarrow \quad x^{(0)} = -2 \quad \text{or} \quad x^{(0)} = -1$$

$$\varepsilon [x^{(0)}]^3 + [x^{(1)}]^2 + 3x^{(1)} + 2 = 0$$

$$\Rightarrow \begin{cases} [x^{(1)}]^2 + 3x^{(1)} + (2-\varepsilon) = 0 \\ \text{or} \\ [x^{(1)}]^2 + 3x^{(1)} + (2-8\varepsilon) = 0 \end{cases}$$

$$\rightarrow \quad x^{(1)} = \frac{-3 \pm \sqrt{9-4(2-\varepsilon)}}{2}$$

$$\text{or} \quad x^{(1)} = \frac{-3 \pm \sqrt{9-4(2-8\varepsilon)}}{2}$$

both suggest an asymptotic expansion of the kind $x = a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \cdots$

$\rightarrow$ Let's use asymptotic series method:

$$x = a_0 + a_1\varepsilon + a_2\varepsilon^2 + \cdots$$

$$\rightarrow \quad \varepsilon(a_0 + a_1\varepsilon + a_2\varepsilon^2 + \cdots)^3 + (a_0 + a_1\varepsilon + a_2\varepsilon^2 + \cdots)^2 + 3(a_0 + a_1\varepsilon + a_2\varepsilon^2 + \cdots) + 2 = 0$$

to 0th order: $\quad a_0^2 + 3a_0 + 2 = 0 \quad \Rightarrow \quad a_0 = -1 \text{ or } a_0 = -2$

to 1st order: $\quad a_0^3 + 2a_0 a_1 + 3a_1 = 0$

$$\Rightarrow \quad a_1 = \frac{-a_0^3}{2a_0+3} = \begin{cases} 1 & \text{if } a_0 = -1 \\ -8 & \text{if } a_0 = -2 \end{cases}$$

$\rightarrow$ The two-term expansions of regular solutions are

$$\begin{cases} x = -1 + \varepsilon \\ x = -2 - 8\varepsilon . \end{cases}$$

For the singular solution we expect x v. large → so to lowest order the terms εx^3 and x^2 must match ⇒

→ $x \sim -\frac{1}{\varepsilon}$ this suggests an asymptotic expansion of the kind $x = \frac{a_{-1}}{\varepsilon} + a_0 + a_1\varepsilon + \dots$

$$\Rightarrow \varepsilon\left(\frac{a_{-1}}{\varepsilon} + a_0 + a_1\varepsilon \dots\right)^3 + \left(\frac{a_{-1}}{\varepsilon} + a_0 + a_1\varepsilon + \dots\right)^2 + 3\left(\frac{a_{-1}}{\varepsilon} + a_0 + a_1\varepsilon + \dots\right) + 2 = 0$$

To lowest order $(O(\varepsilon^{-2}))$: $a_{-1}^3 + a_{-1}^2 = 0 \Rightarrow a_{-1} = -1$ indeed

To next order $(O(\varepsilon^{-1}))$: $3a_{-1}^2 a_0 + 2a_{-1}a_0 + 3a_{-1} = 0$

$$\Rightarrow \quad a_0 = \frac{-3a_{-1}}{3a_{-1}^2 + 2a_{-1}} = \frac{-3}{3a_{-1}+2} = 3$$

→ So the 2-term expansion of the singular term is

$$x = -\frac{1}{\varepsilon} + 3 .$$

<u>Problem 3 p 23</u>. 2-term expansion for $\begin{cases} f'' + f = \varepsilon f' \\ f(0) = 1 \\ f'(0) = 0 \end{cases}$

let's try $f(t) = f_0(t) + \varepsilon f_1(t)$

$$\Rightarrow \quad [f_0'' + \varepsilon f_1''] + f_0 + \varepsilon f_1 = \varepsilon(f_0' + \varepsilon f_1')$$

$$\Rightarrow \quad \begin{cases} f_0'' + f_0 = 0 & f_0(0) = 1 \quad f_0'(0) = 0 \\ f_1'' + f_1 = f_0' & f_1(0) = 0 \quad f_1'(0) = 0 \end{cases}$$

The solution for $f_0(t)$ is $f_0(t) = \cos t$ with these BCs so

$$f_1'' + f_1 = -\sin t$$

$$\Rightarrow \quad f_1(t) = A\cos t + B\sin t + f_{PI} .$$

For the PI we try (by parity)

$f_{PI} = Kt\cos t$ so

$f''_{PI} + f_{PI} = -\sin t \Rightarrow$ $\quad f'_{PI} = K\cos t - Kt\sin t$

$f''_{PI} = -2K\sin t - Kt\cos t$

$-2K\sin t - \cancel{Kt\cos t} + \cancel{Kt\cos t} = -\sin t$

$\Rightarrow K = \frac{1}{2}$ so

$f_1(t) = A\cos t + B\sin t + \frac{1}{2}t\cos t$

To fit the BCs we have to have

$A = 0$ so $f_1(0) = 0$

$B + \frac{1}{2} = 0$ so $f_1'(0) = 0 \rightarrow B = -\frac{1}{2}$

So finally $f(t) = \cos t + \varepsilon\left[-\frac{1}{2}\sin t + \frac{1}{2}t\cos t\right] + o(\varepsilon^2)$

(*) See later for comparison with exact solution

Problem 4 p 23

$$M\frac{d^2X}{dT^2} + \frac{\Lambda}{L}X + K\left(\frac{dX}{dT}\right)^3 = 0$$

let $x = \frac{X}{X_0}$ and $t = T\sqrt{\frac{\Lambda}{ML}}$ $\rightarrow X = xX_0$

$$\frac{d}{dT} = \frac{d}{dt}\frac{dt}{dT} = \sqrt{\frac{\Lambda}{ML}}\frac{d}{dt} \qquad \frac{d}{dX} = \frac{d}{dx}\frac{dx}{dX} = \frac{1}{X_0}\frac{d}{dx}$$

$$\Rightarrow X_0 M \frac{\Lambda}{ML}\frac{d^2x}{dt^2} + X_0\frac{\Lambda}{L}x + KX_0^3\left(\frac{\Lambda}{ML}\right)^{3/2}\left(\frac{dx}{dt}\right)^3 = 0$$

$$\Rightarrow \frac{d^2x}{dt^2} + x + \underbrace{KX_0^2\frac{L}{\Lambda}\left(\frac{\Lambda}{ML}\right)^{3/2}}_{\varepsilon = X_0^2 K M^{-2/3} L^{-1/2} \Lambda^{1/2}}\left(\frac{dx}{dt}\right)^3 = 0$$

At $t=0$ $\begin{cases} X = X_0 \Rightarrow x = \frac{X}{X_0} = 1 \rightarrow x(0) = 1 \\ \frac{dX}{dT} = 0 \Rightarrow \frac{dx}{dt} = 0 \end{cases}$

We then get

$$\frac{d^2x}{dt^2} + x = -\varepsilon\left(\frac{dx}{dt}\right)^3$$

Let's try an expansion of the kind $x(t) = x_0(t) + \varepsilon x_1(t) + \cdots$

→ to lowest order:

$$\frac{d^2x_0}{dt^2} + x_0 = 0 \qquad x_0(t) = A\cos t + B\sin t.$$

BCs are $x_0(0) = 1$, $\frac{dx_0}{dt}(0) = 0 \Rightarrow x_0(t) = \cos t$

→ to next order

$$\frac{d^2x_1}{dt^2} + x_1 = -\left(\frac{dx_0}{dt}\right)^3 = -\left(-\sin t\right)^3 = \sin^3 t = \left(\frac{e^{it} - e^{-it}}{2i}\right)^3$$

$$= \frac{e^{3it} - 3e^{it} + 3e^{-it} - e^{-3it}}{-8i}$$

$$= -\frac{1}{4}\sin 3t + \frac{3}{4}\sin t$$

→ Solutions will be of the kind

$$x_1(t) = \underbrace{A\cos t + B\sin t}_{\text{general sol}} + \underbrace{k_1\sin 3t}_{\text{PI for } \sin 3t \text{ term}} + \underbrace{k_2 t\cos t}_{\text{PI for } \sin t \text{ term.}}$$

$$x_1'(t) = -A\sin t + B\cos t + 3k_1\cos 3t + k_2(\cos t - t\sin t)$$

$$x_1''(t) = -A\cos t - B\sin t - 9k_1\sin 3t + k_2(-2\sin t - t\cos t)$$

$$\Rightarrow \begin{cases} -9k_1\cancel{\sin 3t} + k_1\cancel{\sin 3t} = -\frac{1}{4}\cancel{\sin 3t} & \Rightarrow k_1 = \frac{-\frac{1}{4}}{-8} = \frac{1}{32} \\ -2k_2\cancel{\sin t} = \frac{3}{4}\cancel{\sin t} & \Rightarrow k_2 = -\frac{3}{8} \end{cases}$$

So $x_1(t) = A\cos t + B\sin t + \frac{1}{32}\sin 3t - \frac{3}{8}t\cos t$

For $x_1(0) = 0$ and $\frac{dx_1}{dt}(0) = 0$ we need:

$$A = 0$$

$$B + 3k_1 + k_2 = 0 \quad \Rightarrow \quad B = +\frac{3}{8} - \frac{3}{32} = \frac{9}{32}$$

So finally, $x(t) = \cos t + \varepsilon\left(\frac{9}{32}\sin t + \frac{1}{32}\sin 3t - \frac{3}{8}t\cos t\right) + o(\varepsilon^2)$

(*) Remainder of Problem 3

The exact solution of $f'' - \varepsilon f' + f = 0$ with $\begin{cases} f(0) = 1 \\ f'(0) = 0 \end{cases}$

is found by

- let $f = e^{\lambda t}$ $\Rightarrow$ $\lambda^2 - \varepsilon\lambda + 1 = 0$

$$\Rightarrow \lambda = \frac{\varepsilon \pm \sqrt{\varepsilon^2 - 4}}{2}$$ (assuming small ε)

$$= \frac{\varepsilon \pm 2i\sqrt{1 - \varepsilon^2/4}}{2} = \frac{\varepsilon}{2} \pm i\sqrt{1 - \frac{\varepsilon^2}{4}}$$

- so $f(t) = e^{\frac{\varepsilon}{2}t}\left[A\cos\left(\sqrt{1-\frac{\varepsilon^2}{4}}\,t\right) + B\sin\left(\sqrt{1-\frac{\varepsilon^2}{4}}\,t\right)\right]$

$f(0) = 1 \Rightarrow \quad A = 1$

$f'(0) = 0 \Rightarrow \quad \frac{\varepsilon}{2}A + \sqrt{1 - \frac{\varepsilon^2}{4}}\,B = 0$

$$\Rightarrow \quad B = -\frac{\frac{\varepsilon}{2}}{\sqrt{1-\varepsilon^2/4}}$$

So $f(t) = e^{(\varepsilon/2\, t)}\left[\cos\left(\sqrt{1-\frac{\varepsilon^2}{4}}\,t\right) - \frac{\varepsilon}{2}\frac{1}{\sqrt{1-\varepsilon^2/4}}\sin\left(\sqrt{1-\frac{\varepsilon^2}{4}}\,t\right)\right]$.

Now let's expand this for small εt: $\quad \sqrt{1-\frac{\varepsilon^2}{4}} \simeq 1 - \frac{\varepsilon^2}{8} + \cdots$

$$f(t) \simeq \left(1 + \frac{\varepsilon}{2}t + \ldots\right)\left[\cos t - \frac{\varepsilon}{2}\sin t\right]$$

$$\simeq \cos t + \varepsilon\left[\frac{t}{2}\cos t - \frac{1}{2}\sin t\right].$$ as required!