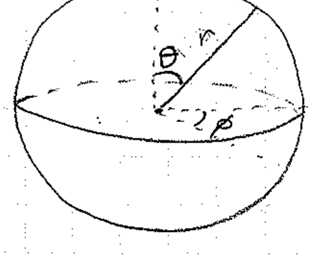
Example of application to 3D problems

Solar oscillations



$0 \le r \le R_\odot$

- The Sun is a spherical ball of self-gravitating gas.
- The outer layer is convecting; the convective motions excite sound waves which propagate throughout the interior. The sound waves satisfy the approximate equation

$$\frac{\partial^2 p}{\partial t^2} = c_s^2(r)\,\nabla^2 p$$

p = pressure

c_s = sound speed, assumed to depend only on radius r

∇^2 = Laplacian in spherical coordinates r, θ, ϕ

$$\Rightarrow \frac{\partial^2 p}{\partial t^2} = c_s^2(r)\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial p}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 p}{\partial\phi^2}\right]$$

We will assume that the boundary conditions are

$$\begin{cases} p(0,\theta,\phi,t) < +\infty \\ p(R_\odot,\theta,\phi,t) = 0 \end{cases}$$

Eigenmodes

Separation of variables $\Rightarrow$ assume

$$p(r,\theta,\phi,t) = A(r)\,B(\theta)\,C(\phi)\,D(t)$$

$$\Rightarrow \underbrace{\frac{1}{D}\frac{d^2D}{dt^2}}_{\text{a function of time only}} = \underbrace{c_s^2(r)\left[\frac{1}{A}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dA}{dr}\right) + \frac{1}{B}\frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dB}{d\theta}\right) + \frac{1}{C}\frac{1}{r^2\sin^2\theta}\frac{d^2C}{d\phi^2}\right]}_{\text{a function of space only}}$$

$$= -\omega^2$$

expect oscillations so select a negative constant

$$\Rightarrow \begin{cases} \dfrac{d^2 D}{dt^2} = -\omega^2 D & \text{①} \\[2ex] \dfrac{1}{A}\dfrac{d}{dr}\left(r^2 \dfrac{dA}{dr}\right) + \dfrac{1}{B\sin\theta}\dfrac{d}{d\theta}\left(\sin\theta \dfrac{dB}{d\theta}\right) + \dfrac{1}{C\sin^2\theta}\dfrac{d^2 C}{d\phi^2} = \dfrac{-\omega^2 r^2}{C_s^2(r)} & \text{②} \end{cases}$$

take ②

$$\Rightarrow \underbrace{\frac{1}{A}\frac{d}{dr}\left(r^2\frac{dA}{dr}\right) + \frac{\omega^2 r^2}{C_s^2(r)}}_{\text{A function of } r \text{ only}} = \underbrace{-\left[\frac{1}{B\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dB}{d\theta}\right) + \frac{1}{C\sin^2\theta}\frac{d^2C}{d\phi^2}\right]}_{\text{A function of } \theta \text{ and } \phi \text{ only.}}$$

$$= -\lambda$$

↑ another constant

so
$$\begin{cases} \dfrac{d}{dr}\left(r^2\dfrac{dA}{dr}\right) + \lambda A = -\dfrac{\omega^2 r^2}{C_s^2(r)} A & \text{③} \\[2ex] \dfrac{1}{B\sin\theta}\dfrac{d}{d\theta}\left(\sin\theta\dfrac{dB}{d\theta}\right) + \dfrac{1}{C\sin^2\theta}\dfrac{d^2C}{d\phi^2} = \lambda & \text{④} \end{cases}$$

take ④, multiply by $\sin^2\theta$ →

$$\underbrace{-\lambda\sin^2\theta + \frac{1}{B}\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{dB}{d\theta}\right)}_{\text{A function of } \theta \text{ only}} = \underbrace{-\frac{1}{C}\frac{d^2C}{d\phi^2}}_{\text{A function of } \phi \text{ only}}$$

$$= m^2$$

↑ a constant.

Since we expect 2π-periodic functions in ϕ, we can straightaway recognize that constant to be m^2, with m integer.

So
$$\begin{cases} \dfrac{d^2C}{d\phi^2} = -m^2 C & ⑤ \\[2ex] \sin\theta \dfrac{d}{d\theta}\left(\sin\theta \dfrac{dB}{d\theta}\right) - m^2 B = \lambda \sin^2\theta\, B & ⑥ \end{cases}$$

$\Rightarrow$ ①, ③, ⑤ and ⑥ are the 4 equations representing the eigenmodes in the t, r, θ and ϕ variables.

<u>Solution in ϕ</u>: naturally $C_m(\phi) = \left\{ \begin{matrix} \cos m\phi \\ \sin m\phi \end{matrix} \right\}$

<u>Solution in θ</u>: The θ-mode depends on the value of m

<u>Trick</u> let $\cos\theta = \mu$
then
$$\frac{d}{d\theta} = \frac{d}{d\mu}\frac{d\mu}{d\theta} = -\sin\theta \frac{d}{d\mu}$$
so ⑥ becomes
$$(1-\mu^2)\frac{d}{d\mu}\left((1-\mu^2)\frac{dB}{d\mu}\right) - m^2 B = \lambda(1-\mu^2) B$$

$$\longrightarrow \quad \frac{d}{d\mu}\left((1-\mu^2)\frac{dB}{d\mu}\right) - \frac{m^2}{1-\mu^2} B = \lambda B$$

The solutions to this equation with $\lambda = -\ell(\ell+1)$ are the Legendre functions
$$\left\{ P_\ell^m(\mu) \right\}_{\ell, m \text{ integers}}$$

So $$B_\ell^m(\theta) = \left\{ P_\ell^m(\cos\theta) \right\}_{m, \ell \text{ integers}}$$

Some properties of Legendre functions:

- $$P_\ell^m(x) = (-1)^{|m|}(1-x^2)^{\frac{|m|}{2}} \frac{d^{|m|} P_\ell(x)}{dx^{|m|}}$$
  where $P_\ell(x)$ is the Legendre Polynomial of order ℓ

- $P_\ell(x) = P_\ell^0(x)$
- $P_0(x) = 1$

  $P_1(x) = x$

$$(n+1) \quad P_{n+1}(x) = (2n+1)x\,P_n(x) - nP_{n-1}(x)$$

- So $P_\ell^m(x) = 0 \quad \forall\ |m| > \ell$

Note: The combination of the two angular functions are usually called <u>spherical harmonics</u> and noted $Y_\ell^m(\theta,\phi)$

e.g. $Y_2^1(\theta,\phi) \propto P_2^1(\cos\theta)\cos\phi$

although standard conventions usually use

$$Y_2^1(\theta,\phi) \propto P_2^1(\cos\theta)e^{i\phi}$$

$$Y_\ell^m(\theta,\phi) \propto P_\ell^m(\cos\theta)e^{im\phi}$$

<u>Solution in r</u>

$$\frac{d}{dr}\left(r^2\frac{dA}{dr}\right) - \ell(\ell+1)A = -\frac{\omega^2 r^2}{c_s^2(r)}A$$

→ clearly, this equation cannot be solved directly without knowledge of $c_s^2(r)$. However, we can already say a lot about the solution by inspection.

+ this is a Sturm-Liouville problem with $p(r) = r^2$, $q(r) = -\ell(\ell+1)$, $w(r) = \frac{r^2}{c_s^2(r)}$, $\lambda = \omega^2$

Special case

Let's take the simplest possible example, that of constant sound-speed $c_s(r)$. Then

$$\frac{d}{dr}\left(r^2 \frac{dA}{dr}\right) - \ell(\ell+1)A = -\frac{\omega^2}{c_s^2} r^2 A$$

This is actually the equation for a spherical Bessel function. Indeed, spherical Bessel functions satisfy

$$\frac{d}{dx}\left(x^2 \frac{dA}{dx}\right) - \ell(\ell+1)A + x^2 A = 0. \quad \Rightarrow \text{ as above if } x = \frac{\omega}{c} r$$

$$A_\ell(x) = \begin{Bmatrix} j_\ell(x) \\ y_\ell(x) \end{Bmatrix}.$$

The $y_\ell(x)$ functions are singular at $x=0$ $\rightarrow$ discard.

At the surface, $A_\ell(x) = 0 \Rightarrow A_\ell\left(\frac{\omega}{c} R_*\right) = 0$.

$\Rightarrow$ this means that $\frac{\omega}{c} R_*$ are zeros of the Bessel $j_\ell(x)$ function. There is an infinite number of them, noted $z_{\ell n}$ (the n-th zero of the $j_\ell(x)$ function)

$$\Rightarrow \quad \omega_{n\ell} = \frac{z_{\ell n}}{R_*} c_s.$$

$\Rightarrow$ Finally, putting everything together we have

$$p(r,\theta,\phi,t) = \sum_{n,\ell,m} A_{\ell n}(r) B_\ell^m(\theta) C_m(\phi) D_{n\ell m}(t) \quad \text{where}$$

$$A_{\ell n}(r) = j_\ell\left(z_{\ell n} \frac{r}{R_*}\right)$$

$$C_m(\phi) B_\ell^m(\theta) = Y_\ell^m(\theta,\phi) \quad \leftarrow \text{ spherical harmonic}$$

$$D_{n\ell m}(t) = a_{n\ell m} \cos(\omega_{\ell n} t) + b_{n\ell m} \sin(\omega_{\ell n} t)$$

where $\omega_{\ell n} = \frac{z_{\ell n}}{R_*} c_s$

In reality, however, $c(r)$ is not constant, which makes the problem hard to solve analytically.

<u>However</u>: • Using asymptotic theory we know that the eigenvalues of the SL problem in radius must satisfy, for fixed ℓ and large n:

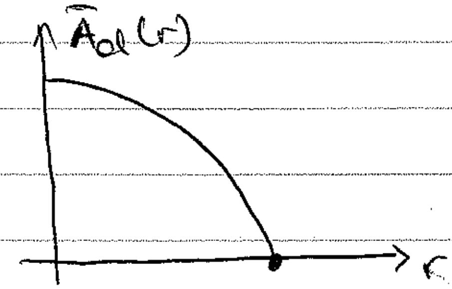
$$\omega_{n\ell}^2 = \left(\frac{n\pi}{\int_0^R \sqrt{\frac{r^2/c^2}{r^2}}\, dr} \right)^2 = \left(\frac{n\pi}{\int_0^R \frac{dr}{c(r)}} \right)^2$$

$\int_0^R \frac{dr}{c(r)}$ = time it takes for sound to cross the entire star so

$$\omega_{n\ell} = \frac{n\pi}{T} \quad \text{where } T \text{ is sound-crossing time}$$

→ we see that the Sun "rings" like a musical instrument in high n modes.

• Using Rayleigh Quotient, we can estimate eigenvalues for low-n modes. For instance, for $n=0$, let's guess that the eigenmode looks like



$\bar{A}_{0\ell}(r)$ is guess function.

→ a good approximation to $\omega_{0\ell}$ is

$$\omega_{0\ell} \simeq - \frac{\int_0^R \bar{A}_{0\ell} \left[\frac{d}{dr}\left(r^2 \frac{dA_{0\ell}}{dr}\right) - \ell(\ell+1) A_{0\ell} \right] dr}{\int_0^R \frac{r^2}{c^2} \cdot A_{0\ell}^2\, dr}.$$

$$\Rightarrow \omega_{0\ell} \cong \frac{0 + \int_0^R r^2 \left(\frac{dA_{0\ell}}{dr}\right)^2 dr + \ell(\ell+1)\int_0^R A_{0\ell}^2\, dr}{\int_0^R \frac{r^2}{c^2(r)} A_{0\ell}^2\, dr}$$

We could plug this into Wolfram and/or solve it numerically or analytically for any input $\bar{A}_{0\ell}$ and $c^2(r)$, to infer $\omega_{0\ell}$. However, we can actually already say more just by inspection.

Indeed, we find that for large ℓ, then

$$\omega_{0\ell} \simeq \ell(\ell+1) \frac{\int_0^R \bar{A}_{0\ell}^2\, dr}{\int_0^R \frac{r^2}{c^2(r)} A_{0\ell}^2\, dr}$$

$= \ell.(\ell+1)$ • A number that only depends on some integral of c^2 over the star.

Furthermore, if we repeat this for any n (not necessarily $n=0$), we would get (using other guess functions with multiple zeros) that

$\omega_{n\ell} \simeq \ell.(\ell+1)$. A number that only depends on the function c^2 and n.

<u>This has several consequences</u>

- For fixed n, and a given star (so fixed $c^2(r)$), the frequencies vary like

  $\omega_{n\ell} \propto \ell(\ell+1)$

  → This can easily be seen in the data! (show data)

- By studying the prefactors $\omega_{nl} = W_n\, l(l+1)$ we can actually infer what $c^2(r)$ is

→ This is what helioseismology effectively does!