(I) Introduction

We now consider forced linear PDES of the kind

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = F(x,t) \qquad \text{(Forced Wave Equation)}$$

$$\frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} = F(x,t) \qquad \text{(Forced heat Equation)}$$

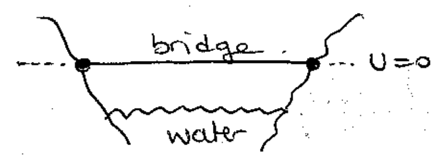
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = F(x,y) \qquad \text{("Forced Laplace equation" = Poisson equation)}$$

The method for solving these problems will be first illustrated through examples, then generalized in (II)

(1) Forced wave equation

Example : A bridge, suspended, and the wind forcing (cf Tacoma Narrows)
(in 2D : a metal plate, with some sand on it, and a speakerphone nearby; see Exploratorium).

let
$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = F(x,t) & \\ u(x, t=0) = 0 & u(0,t) = 0 \\ u_t(x, t=0) = 0 & u(L,t) = 0 \end{cases}$$



→ A forced string, pinned at the sides, initially at rest.

Method: 1. Find the spatial eigenmodes $A_n(x)$ homogeneous problem - with same bcs

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 .$$

These will generally be mixtures of sines and cosines.

Here (see previous lectures)

$$A_n(x) = \sin\left(\frac{n\pi x}{L}\right) .$$

2. Assume that the full solution can be written as

$$u(x,t) = \sum_{n=0}^{\infty} A_n(x) B_n(t)$$

and plug into PDE

$$\Rightarrow \sum_{n=0}^{\infty} A_n(x) \frac{d^2 B_n}{dt^2} - c^2 B_n(t) \frac{d^2 A_n}{dx^2} = F(x,t)$$

$$\Rightarrow \sum_{n=0}^{\infty} A_n(x) \frac{d^2 B_n}{dt^2} + \frac{c^2 n^2 \pi^2}{L^2} B_n(t) A_n(x) = F(x,t). \quad (*)$$

3. Note that $\int_0^L A_n(x) A_m(x)\, dx = 0 \quad \forall n \neq m$

$$= \frac{L}{2} \text{ if } n=m$$

<u>Aside</u>: The orthogonality property of the eigenfunctions will be true for a wide class of problems, see next chapter.

So take (*) and multiply by $A_m(x)$, then integrate over $[0, L]$

$$\Rightarrow \frac{L}{2} \frac{d^2 B_m}{dt^2} + \frac{c^2 m^2 \pi^2}{L^2} \cdot \frac{L}{2} B_m = \int_0^L F(x,t) \sin\left(\frac{m\pi x}{L}\right) dx.$$

$$\Rightarrow \ddot{B}_m + \frac{c^2 m^2 \pi^2}{L^2} B_m = \frac{2}{L} \int_0^L F(x,t) \sin\left(\frac{m\pi x}{L}\right) dx$$

$$= f_m(t).$$

$\Rightarrow$ we now get a set of independent ODES, one for each value of m. These are forced, second-order linear ODES. (which you should be able to solve...)

Simple example

Suppose $F(x,t) = \sin\left(\frac{2\pi x}{L}\right)\cos(\omega t)$

then $f_m(t) = \int_0^L \frac{2}{L} \sin\left(\frac{2\pi x}{L}\right)\cos(\omega t)\sin\left(\frac{m\pi x}{L}\right)dx$

$$= \begin{cases} \frac{2}{L} \cdot \frac{L}{2} \cdot \cos\omega t & \text{if } m=2 \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow$ we have 2 types of ODES to solve:

$$\ddot{B}_2 + \frac{4c^2\pi^2}{L^2} B_2 = \cos\omega t$$

and $$\ddot{B}_m + \frac{c^2m^2\pi^2}{L^2} B_m = 0 \qquad \forall\, m \neq 2.$$

For all of them, the solution to the homogeneous equation is

$$B_m(t) = \alpha_m \cos\left(\frac{cm\pi}{L} t\right) + \beta_m \sin\left(\frac{m\pi ct}{L}\right)$$

For the $B_2(t)$ function, we have to add a particular solution to the forced problem: here try

$$B_2^{PS}(t) = K \cos\omega t$$

a constant to be determined by plugging into equation

$$\Rightarrow \quad -K\omega^2 + \frac{4c^2\pi^2}{L^2} K = 1$$

$$\Rightarrow \quad K = \frac{1}{\frac{4c^2\pi^2}{L^2} - \omega^2}$$

$$\Rightarrow \quad B_2(t) = \alpha_2 \cos\left(\frac{2c\pi t}{L}\right) + \beta_2 \sin\left(\frac{2c\pi t}{L}\right) + \frac{\cos(\omega t)}{\frac{4c^2\pi^2}{L^2} - \omega}$$

$\Rightarrow$ So finally, we have

$$u(x,t) = \sum_{n=1}^{\infty} A_n(x)\, B_n(t) \quad \text{where} \quad A_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

$$B_n(t) = \alpha_n \cos\left(\frac{n\pi ct}{L}\right) + \beta_n \sin\left(\frac{n\pi ct}{L}\right) + \frac{\cos \omega t}{\frac{4c^2\pi^2}{L^2} - \omega^2}\, \delta_{n,2}$$

To find the arbitrary constants α_n, β_n, we fit the initial conditions:

$u(x,0) = 0$

$$\Rightarrow \sum_{n=1}^{\infty} A_n(x) B_n(0) = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} A_n(x) \left[\alpha_n + \frac{1}{\frac{4c^2\pi^2}{L^2} - \omega^2}\, \delta_{n,2} \right] = 0$$

$$\Rightarrow \alpha_n + \frac{1}{\frac{4c^2\pi^2}{L^2} - \omega^2}\, \delta_{n,2} = 0$$

$$\Rightarrow \begin{cases} \alpha_n = 0 & \text{if } n \neq 2 \\ \alpha_2 = -\dfrac{1}{\frac{4c^2\pi^2}{L^2} - \omega^2} \end{cases}$$

$$u_t(x,0) = 0 \Rightarrow \sum_{n=1}^{\infty} A_n(x)\, \dot{B}_n(0) = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} A_n(x) \left[\beta_n \frac{n\pi c}{L} \right] = 0$$

$$\Rightarrow \beta_n = 0 \quad \forall n.$$

So: the solution becomes very simple!

$$u(x,t) = \left[- \frac{1}{\frac{4c^2\pi^2}{L^2} - \omega^2} \cdot \cos\left(\frac{2\pi c t}{L}\right) + \frac{\cos\omega t}{\frac{4c^2\pi^2}{L^2} - \omega^2} \right] \sin\left(\frac{2\pi x}{L}\right)$$

<u>We note</u>: • $F(x,t) = \cos\omega t \, \sin\left(\frac{2\pi x}{L}\right)$

specifically forces the system in one of its <u>spatial eigenmodes</u>

→ then this eigenmode is the only one to be excited (see solution).

If $F(x,t)$ had a more complex spatial structure, other modes would be excited too.

• The solution responds by oscillating at <u>two</u> different frequencies simultaneously:

Beating phenomenon. ⇐ { – at the <u>forcing</u> frequency
– at the <u>intrinsic</u> frequency of the eigenmode.

• The amplitude of the response goes like

$$\frac{1}{\underbrace{\frac{4c^2\pi^2}{L^2}}_{\text{intrinsic frequency squared}} - \underbrace{\omega^2}_{\text{forcing frequency square}}}$$

→ if the forcing frequency approaches the intrinsic frequency then the amplitude of the response can be <u>huge</u>

This phenomenon is called <u>resonnance</u>.

<u>Note</u>: That doesn't mean the amplitude can ever be ∞: instead if $\omega = \frac{4c^2\pi^2}{L^2}$ the amplitude of the mode grows linearly with time. (Homework: prove this).