(6) An eigenvalue λ is a stationary point of the Rayleigh quotient, and is equal to the Rayleigh quotient applied to the corresponding eigenfunction.

$$\lambda = \frac{\iiint_D u_\lambda \left[\nabla^2 u_\lambda\right] d^3\underline{r}}{\iiint_D u_\lambda^2 \, d^3\underline{r}}$$

$$= \frac{-\iiint_D |\nabla u_\lambda|^2 d^3\underline{r} + \iint_{\partial D} u_\lambda \nabla u_\lambda \cdot \underline{n} \, dS}{\iiint_D u_\lambda^2 \, d^3\underline{r}}$$

(by integration by part).

Note that, with care, these theorems can be applied to 1D problems although one must be extremely careful about what the integral means

e.g. if it is a "1D" problem that arises from a 3D Cartesian problem which happens to be invariant in y & z, for example, then

$$\int d^3\underline{r} \to \int dx\,dy\,dz \to \int dx.$$

However if it is a "1D" problem that arises from a 3D spherical problem that happens to be invariant in θ, ϕ say then

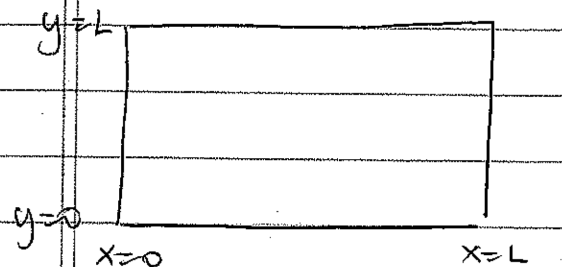
$$\int d^3\underline{r} \to \int r^2 dr \sin\theta \, d\theta \, d\phi \to \int r^2 dr$$

This r^2 term is equivalent to the weight function we would have deduced from working with a 1D problem to start with.

This generalization to higher dimensions also explains why some of the singular SL problems in 2D (specifically, the ones that arise from coordinate singularities) have many of the properties of the regular SL problems. These properties are the ones listed as ① - ⑥ in this section.

II Example of the 2D rectangle cooling problem

Let's consider a 2D conducting metal plate, of size
$x \in [0, L]$ and $y \in [0, H]$



The sides are held at $T=0$
The initial temperature profile of the plate is $T_0(x,y)$
What is $T(x,y,t)$?

$\Rightarrow$ Equation of evolution is
$$\frac{\partial T}{\partial t} = k \nabla^2 T = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

Let's first assume solutions are separable in space & time: $T(x,y,t) = A(x,y)\, B(t) \Rightarrow$

$$\begin{cases} \dfrac{dB}{dt} = -k\lambda B \\ \nabla^2 A = -\lambda A = \dfrac{\partial^2 A}{\partial x^2} + \dfrac{\partial^2 A}{\partial y^2}. \end{cases}$$

We can then also seek solutions for A that are separable in x & y:
$$A(x,y) = a(x)\, b(y)$$

This implies

$$b\frac{d^2a}{dx^2} + a\frac{d^2b}{dy^2} = -\lambda ab$$

$$\Rightarrow \frac{1}{a}\frac{d^2a}{dx^2} + \frac{1}{b}\frac{d^2b}{dy^2} = -\lambda$$

$$\Rightarrow \frac{1}{a}\frac{d^2a}{dx^2} = -\lambda - \frac{1}{b}\frac{d^2b}{dy^2} = -\gamma$$

$$\Rightarrow \begin{cases} \dfrac{d^2a}{dx^2} = -\gamma a \\[2ex] \dfrac{d^2b}{dy^2} = -(\lambda-\gamma)\,b \end{cases}$$

- Applying bcs in the x- direction we find that

$$a(x) = \alpha\cos(\sqrt{\gamma}\,x) + \beta\sin(\sqrt{\gamma}\,x)$$

$$a(0)=0 \Rightarrow \alpha = 0$$

$$a(L)=0 \Rightarrow \sqrt{\gamma} = \frac{n\pi}{L} \Rightarrow \begin{cases} \gamma_n = \dfrac{n^2\pi^2}{L^2} \\[2ex] a_n(x) = \sin\left(\dfrac{n\pi x}{L}\right) \end{cases}$$

- Applying bcs in the y- direction,

$$b(x) = \alpha\cos\left(\sqrt{\lambda-\gamma}\,y\right) + \beta\sin\left(\sqrt{\lambda-\gamma}\,y\right)$$

$$b(0)=0 \Rightarrow \alpha = 0$$

$$b(H)=0 \Rightarrow \sqrt{\lambda-\gamma} = \frac{m\pi}{H} \qquad \begin{cases} \lambda_{mn} = \gamma_n + \dfrac{m^2\pi^2}{H^2} \\ \quad \uparrow \ \lambda \text{ now carries } \underline{\underline{2}} \text{ indices.} \\[2ex] b_m(y) = \sin\left(\dfrac{m\pi y}{H}\right) \end{cases}$$

$\Rightarrow$ The spatial 2D eigenfunction is now characterized by 2 indices:

$$\begin{cases} A_{nm}(x,y) = a_n(x)\,b_m(y) \\ \lambda_{nm} = \dfrac{n^2\pi^2}{L^2} + \dfrac{m^2\pi^2}{H^2} \end{cases}$$

$\rightarrow$ For each of these there is one temporal eigenfunction
$B_{mn}(t) = e^{-\lambda_{mn} kt}$

$\Rightarrow$ General solution:

$$T(x,y,t) = \sum_{m,n} \alpha_{mn} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{L}\right) e^{-\left(\frac{n^2\pi^2}{L^2} + \frac{m^2\pi^2}{H^2}\right)kt}.$$

Note: One can see that

- the eigenvalues λ_{mn} are real
- under certain circumstances there can be more than one A_{nm} eigenfunction associated to the same eigenvalue (i.e. whenever $\exists$ 2 pairs (n_1, m_1) and (n_2, m_2) s.t $\frac{n_1^2}{L^2} + \frac{m_1^2}{H^2} = \frac{n_2^2}{L^2} + \frac{m_2^2}{H^2}$)
- The eigenfunctions as defined are indeed orthogonal wrt the inner product

$$\langle u_1, u_2 \rangle = \int_0^L\!\!\int_0^H u_1(x,y)\, u_2(x,y)\, dx\, dy.$$

$\Rightarrow$ To find the coefficients α_{nm} of the final solutions, we have

$$T_0(x,y) = \sum_{m,n} \alpha_{mn} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right).$$

so

$$\alpha_{mn} = \frac{\iint T_0(x,y) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right) dx\, dy}{\iint \sin^2\left(\frac{n\pi x}{L}\right) \sin^2\left(\frac{m\pi y}{H}\right) dx\, dy}.$$