(See Bessel handout)

This equations "looks" like the Bessel equation for $n=0$, but isn't quite

Bessel eq: $\quad x^2 f'' + x f' + (x^2 - n^2) f = 0$

For $n=0$

$$x^2 f'' + x f' + x^2 f = 0$$

while we have

$$x^2 A'' + x A' + \lambda x^2 A = 0$$

However, note that if rescale x so that

$$\hat{x}^2 = \lambda x^2 \qquad (\text{i.e introduce } \hat{x} = \sqrt{\lambda}\, x)$$

then, when written in terms of the new variable $\hat{x}$,

$$x^2 A'' + x A' + \lambda x^2 A = 0$$

$$\Rightarrow \quad \hat{x}^2 \ddot{A} + \hat{x} \dot{A} + \hat{x}^2 A = 0 \quad \leftarrow \text{A Bessel eq. for } n=0.$$

So the solutions are

$$A(\hat{x}) = \begin{Bmatrix} J_0(\hat{x}) \\ Y_0(\hat{x}) \end{Bmatrix} \rightarrow A(x) = \begin{Bmatrix} J_0(\sqrt{\lambda}\, x) \\ Y_0(\sqrt{\lambda}\, x) \end{Bmatrix}$$

J_0 is regular at $\hat{x}=0$ while Y_0 is singular at $x=0$. To keep the problem physically meaningful, we can ignore the singular solution so the solution to our problem is $\quad A(x) = J_0(\sqrt{\lambda}\, x)$

The BC at $x=R$ implies $A(R)=0$

$$\Rightarrow \quad J_0(\sqrt{\lambda}\, R) = 0.$$

This determines the eigenvalues λ_n, since there are many possible solutions.

The function $J_0(x)$ has many zeros, listed as $z_0, z_1, z_2, \dots z_n$.

$$\rightarrow \quad \sqrt{\lambda_n}\, R = z_n \Rightarrow \lambda_n = \left(\frac{z_n}{R}\right)^2$$

So finally, the family of eigenfunctions are

$$A_n(x) = J_0\left(\frac{z_n}{R} x\right) \quad \text{with eigenvalue } \lambda_n$$

where z_n is a zero of the Bessel function J_0.

Temporal problem : $\frac{d^2 B_n}{dt^2} = -c^2 \lambda_n B_n$

So $B_n(t) = \alpha_n \cos\left(\sqrt{\lambda_n}\, ct\right) + \beta_n \sin\left(\sqrt{\lambda_n}\, ct\right)$.

where $\sqrt{\lambda_n} = z_n/R$

$\Rightarrow$ General solution :

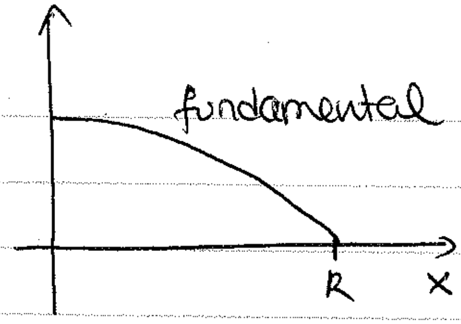
$$u(x,t) = \sum_n \left(\alpha_n \cos\left(\frac{z_n}{R} ct\right) + \beta_n \sin\left(\frac{z_n}{R} ct\right)\right) J_0\left(\frac{z_n}{R} x\right)$$

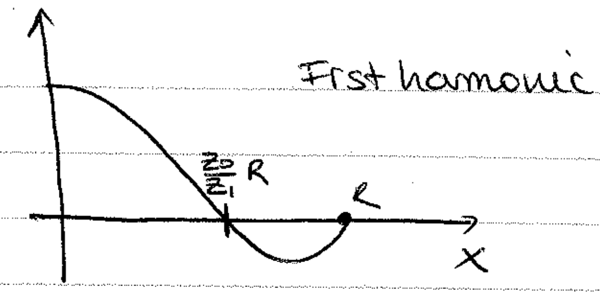
Discussion : As in the case of the vibrating string we find that each spatial $J_0\left(\frac{z_n}{R}x\right)$ mode oscillates with its own eigenfrequency $\nu_n = \frac{z_n c}{R}$ ... nothing very new here.

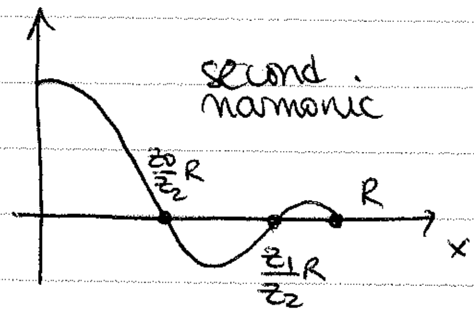
What does SL theory tell us?

Note : We need to be very careful here since this is a singular problem. However, many of the properties of regular problems remain, as this is a regular-singular problem (i.e supports regular & singular solutions).

- The number of nodes of $A_n(x)$ in $(0, R)$ is n:







etc...

- The fundamental eigenvalue can be approximated via the Rayleigh quotient

Here,
$$R(u) = -\frac{\int_0^R u\left[(xu_x)_x + 0\right]dx}{\int_0^R x u^2\,dx}$$

$$= \frac{-\cancel{\left[u x u_x\right]_0^R} + \int_0^R x u_x^2\,dx}{\int_0^R x u^2}$$

$$= \frac{\int_0^R x u_x^2\,dx}{\int_0^R x u^2\,dx.}$$

As a simple example let's take $u_0 = 1 - \left(\frac{x}{R}\right)^2$ as a function which satisfies the BCs...
$\rightarrow$ an approximation to λ_0 is

$$R(u_0) = \frac{\int_0^R x\left(-\frac{2x}{R^2}\right)^2 dx}{\int_0^R x\left(1 - 2\frac{x^2}{R^2} + \frac{x^4}{R^4}\right)dx} = \frac{\frac{4}{4}\frac{R^4}{R^4}}{\frac{R^2}{2} - \frac{2R^4}{4R^2} + \frac{R^6}{6R^4}} = \frac{6}{R^2}$$

Meanwhile, $\lambda_0 = \frac{z_0^2}{R^2}$ with $z_0 \cong 2.4$

$\simeq \frac{5.76}{R^2}$. Not perfect, but really pretty good approximation.

- $A_n(x)$ can be approximated for large n <u>and</u> x not close to 0.

$$A_n(x) = \frac{1}{(x^2)^{1/4}} \left\{ \alpha \cos\left[\sqrt{\lambda_n} \int_0^x dx\right] + \beta \sin\left[\sqrt{\lambda_n} \int_0^x dx'\right] \right\}$$

since $r(x) = p(x) = x$, where

$$\lambda_n \simeq \left(\frac{n\pi}{R}\right)^2$$

$$\rightarrow A_n(x) \simeq \frac{1}{x^{1/2}} \left\{ \alpha \cos\left(\frac{n\pi x}{R}\right) + \beta \sin\left(\frac{n\pi x}{R}\right) \right\}$$

To guarantee regularity @ $x=0$ and satisfy the BCs at $x=R$,

$$A_n(x) = \frac{1}{\sqrt{x}} \sin\left(\frac{n\pi x}{R}\right).$$

(see Maple file)

We see that the mode is well approximated by $\frac{1}{\sqrt{x}} \sin\left(\frac{n\pi x}{R}\right)$ (modulo an amplitude) except near $x=0$ $\rightarrow$ discrepancy comes from the fact that WKB theory fails if $p(x)$ or $r(x) \rightarrow 0$.

- Orthogonality of the eigenfunctions & applying the ICs

The S.L theory tells us that $\neq$ eigenfunctions are orthogonal w.r.t the inner product

$$\langle u, v \rangle = \int_0^R x\, u(x) v(x)\, dx$$

$\Rightarrow$ This means that $\forall\, n \neq m$, we have

$$\int_0^R x\, J_0\left(\frac{z_n}{R}x\right) J_0\left(\frac{z_m}{R}x\right) dx = 0.$$

This property can then be used to construct the final solution by applying the ICs:

$$u_t(x,0) = e^{-x^2/2\sigma^2} = \sum_n{}' \frac{z_n}{R} c\, \beta_n\, J_0\left(\frac{z_n}{R}x\right)$$

If we multiply by x and $J_0\left(\frac{z_m}{R}x\right)$ on both sides & integrate over $[0, R]$, we get

$$\int_0^R x e^{-\frac{x^2}{2\sigma^2}} J_0\left(\frac{z_m}{R}x\right) dx = \frac{z_m}{R} c\, \beta_m \int_0^R x \left[J_0\left(\frac{z_m}{R}x\right)\right]^2 dx$$

(all other terms vanish)

$$\Rightarrow \quad \beta_m = \frac{R}{c z_m} \frac{\int_0^R x e^{-\frac{x^2}{2\sigma^2}} J_0\left(\frac{z_m}{R}x\right) dx}{\int_0^R x \left[J_0\left(\frac{z_m}{R}x\right)\right]^2 dx}$$

$\rightarrow$ Can be evaluated numerically, for example. (see movie)

CHAPTER 6 Higher-dimensional PDEs

In this Chapter we will study how to use the method of separation of variables when applied to problems with more than 2 independent variables e.g. (t,x,y) or (x,y,z), or (r,θ,ϕ,t), etc.
As we will see the method is essentially the same but requires more book-keeping.

I General considerations

Let's consider the following types of problems

1 • $\begin{cases} \frac{\partial u}{\partial t} = k\nabla^2 u \\ \text{for } \underline{r} \in D \end{cases}$ with u having homogeneous BCs (of any kind) on ∂D

2 • $\begin{cases} \frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u \\ \text{for } r \in D \end{cases}$ with u " " "
" " " "

Not considered in following examples but behaving in similar way.
$\left[\bullet \begin{cases} \nabla^2 u = 0 \\ \text{for } r\in D \end{cases}\right.$ with u having homogeneous BCs on at least 2 of the 3 independent variables.$\Big]$

Applying separation of variables between spatial & temporal variables we assume $u(\underline{r},t) = A(\underline{r})B(t)$
so that

1 $\begin{cases} \frac{dB}{dt} = -\lambda k B \\ \nabla^2 A = -\lambda A \end{cases}$ 2 $\begin{cases} \frac{d^2B}{dt^2} = -\lambda c^2 B \\ \nabla^2 A = -\lambda A. \end{cases}$

→ This time, the spatial problem reduces to what is commonly known as the <u>Helmholtz equation</u>, which is a special case of

$$\boxed{\nabla\cdot(p\nabla u) + qu = -\lambda w u} \qquad \left(\begin{matrix}p, q, w \\ \text{depend on } \underline{r}\end{matrix}\right)$$

with $p = w = 1$, $q = 0$.

This form can be recognized fairly easily as the multi-D version of a SL problem.

Many of the theorems concerning SL problems still apply, albeit in a more moderate form in some cases:

① All eigenvalues are real

② Unlike regular SL problems, a given eigenvalue can have more than one eigenfunction

③ There exists an ∞ number of eigenvalues λ_n, starting from the fundamental (i.e. the smallest non-zero eigenvalue) & increasing with n (but not necessarily strictly increasing)

④ The eigenfunctions form a complete set, so that any piecewise smooth function can be written as $f(\underline{r}) = \sum_{\lambda} \alpha_\lambda u_\lambda(\underline{r})$

Note that the weight function here is always 1.

⑤ Eigenfunctions corresponding to $\neq$ eigenvalues are orthogonal wrt the inner product

$$\langle u_1, u_2\rangle = \iiint_D u_1 u_2 \, d^3\underline{r}$$

If they correspond to the <u>same</u> eigenvalue λ, then they can be made to be orthogonal via Gram-Schmidt orthogonalization process.