3.3.2 Parabolic equations

To transform a parabolic equation into its canonical form, we require a change of coordinate acting such that

$$B = C = 0 \quad \text{(in the notation of 3.2)}$$

However since by definition $AC - B^2 = 0$, it is sufficient to require that $C = 0$

$\Rightarrow$ we need $\quad a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = 0$

But now recall that $ac - b^2 = 0$ so this is a perfect square so that it can be rewritten as

$$a\left(\eta_x + \sqrt{\frac{c}{a}}\,\eta_y\right)^2 = 0$$

alternatively $\quad \frac{1}{a}\left(a\eta_x + b\eta_y\right)^2 = 0.$

$\Rightarrow$ we can take η solution of the first order PDE

$$a\eta_x + b\eta_y = 0$$

$\Rightarrow$ η constant on the characteristics defined by $\frac{dy}{dx} = \frac{b}{a}$

[Note that this time ξ can be any function of x and y such that the Jacobian of (ξ, η) doesn't vanish

<u>Example</u>: $\quad x^2 u_{xx} - 2xy\, u_{yx} + y^2 u_{yy} + x u_x + y u_y = 0$

$$\delta(\mathcal{L}) = x^2y^2 - x^2y^2 = 0$$

The characteristics satisfy $\quad \frac{dy}{dx} = \frac{xy}{x^2} = -\frac{y}{x}$

So $\quad \ln y = -\ln x + \text{const}$

or $\quad y = \frac{K}{x} \quad \Rightarrow$ take $\eta = xy$ and for simplicity, $\xi = x$

$$u_x = u_\xi + y u_\eta = v_\xi + \frac{\eta}{\xi} v_\eta \quad \text{since} \quad \xi_x = 1 \quad \xi_y = 0$$

$$u_y = x u_\eta = \xi v_\eta \qquad \eta_x = y \quad \eta_y = x$$

$$u_{xx} = u_{\xi\xi} + y u_{\eta\xi} + y^2 u_{\eta\eta} = v_{\xi\xi} + \frac{\eta}{\xi} v_{\eta\xi} + \left(\frac{\eta}{\xi}\right)^2 v_{\eta\eta}$$

$$u_{xy} = xy\, u_{\eta\eta} + x\, u_{\eta\xi} + u_\eta = \eta v_{\eta\eta} + \xi v_{\eta\xi} + v_\eta$$

$$u_{yy} = x^2 u_{\eta\eta} = \xi^2 v_{\eta\eta}$$

So we now have

$$\xi^2 \left[u_{\xi\xi} + 2\frac{\eta}{\xi} u_{\eta\xi} + \frac{\eta^2}{\xi^2} u_{\eta\eta} \right]$$

$$-2\eta \left[\eta\, u_{\eta\eta} + \xi u_{\eta\xi} + u_\eta \right]$$

$$+ \frac{\eta^2}{\xi^2} \left[\xi^2 u_{\eta\eta} \right] + \xi \left(u_\xi + \frac{\eta}{\xi} u_\eta \right) + \frac{\eta}{\xi} \left(\xi u_\eta \right) = 0$$

$$\Leftrightarrow \quad \xi^2 u_{\xi\xi} + \xi u_\xi = 0 \quad \Leftrightarrow \quad \boxed{u_{\xi\xi} + \frac{1}{\xi} u_\xi = 0}$$

→ the canonical form required.

This is now a simple ODE for $v = \frac{\partial u}{\partial \xi}$:

$$v_\xi + \frac{1}{\xi} v = 0$$

$$\Rightarrow \quad \frac{dv}{v} = -\frac{1}{\xi} d\xi \quad \rightarrow \quad \ln v = -\ln \xi + f(\eta)$$

$$\rightarrow \quad v = \frac{\hat{f}(\eta)}{\xi}$$

then $u_\xi = \frac{\hat{f}(\eta)}{\xi} \Rightarrow u(\xi, \eta) = \ln \xi \cdot \hat{f}(\eta) + g(\eta)$

$$\Rightarrow \quad u(x, y) = f(xy) \cdot \ln x + g(xy).$$

3.3.3 Canonical form for Elliptic equations

Given a second order linear PDE which is elliptic, to reduce it to its canonical form we must find a coordinate change $(x,y) \to (\xi,\eta)$ such that

$$\begin{cases} A = C \\ B = 0 \end{cases}$$

So we need
$$\begin{cases} a\xi_x^2 + 2b\xi_x\xi_y + c\,\xi_y^2 = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 \\ a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y = 0 \end{cases}$$

let's construct the complex quantity $\phi = \xi + i\eta$ then this system is equivalent to

$$a\phi_x^2 + 2b\phi_x\phi_y + c\phi_y^2 = 0$$

Indeed

$$a\phi_x^2 + 2b\phi_x\phi_y + c\phi_y^2 = a(\xi_x + i\eta_x)^2 + 2b(\xi_x + i\eta_x)(\xi_y + i\eta_y) + c(\xi_y + i\eta_y)^2$$
$$= a\xi_x^2 - a\eta_x^2 + 2b(\xi_x\xi_y - 2b\eta_x\eta_y + c\xi_y^2 - c\eta_y^2 + i\left[2a\xi_x\eta_x + 2b(\xi_x\eta_y + \eta_x\xi_y) + 2c\,\xi_y\eta_y\right]$$

So equating real & imaginary parts to 0 recovers the required system.

$\Rightarrow$ Characteristic equations imply
$$\frac{dy}{dx} = \frac{b \pm i\sqrt{ac-b^2}}{a} \qquad \text{since } ac - b^2 < 0$$
however, this time the characteristics "live" in a "complex plane".

The characteristic equations are complex conjugates so their solutions (say ϕ and ψ) will also be C.Cs.

Once the solution is found, we recover ξ and η by taking

$$\xi = Re(\phi)$$
$$\eta = Im(\phi).$$

(Note: we can arbitrarily choose ϕ or ψ $\rightarrow$ the only difference is in the sign of η).

<u>Example</u>: the Tricomi equation $u_{xx} + x u_{yy} = 0$ for $x > 0$

then we solve

$$\frac{dy}{dx} = \pm i\sqrt{x} \implies dy = \pm i\sqrt{x}\, dx$$

so the solution is

$$\frac{3}{2} y = \pm i x^{3/2} + \text{constant} \quad \rightarrow \text{choose constant} = \phi$$

so let $\phi = \frac{3}{2} y \pm i x^{3/2}$

so
$$\begin{cases} \xi = \frac{3}{2} y \\ \eta = x^{3/2} \end{cases}$$

then
$$\begin{cases} \xi_x = 0 & \xi_y = \frac{3}{2} \\ \eta_x = \frac{3}{2} x^{1/2} & \eta_y = 0 \qquad \eta_{xx} = \frac{3}{4} x^{-1/2} \end{cases}$$

so
$$u_{xx} + x u_{yy} = \frac{9}{4} x\, u_{\eta\eta} + \frac{3}{4} x^{-1/2} u_\eta + x\left(\frac{9}{4} u_{\xi\xi}\right) = 0$$

$$x = \left(\frac{2}{3}\eta_x\right)^2$$

$$\implies u_{\eta\eta} + u_{\xi\xi} + \frac{1}{3} x^{-3/2} u_\eta = 0$$

$$\implies u_{\xi\xi} + u_{\eta\eta} + \frac{1}{3\eta} u_\eta = 0 \quad \rightarrow \text{Canonical form of the equation for } x > 0$$

SUMMARY

When trying to find the canonical form of

$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} + \mathcal{L}^{(1)}(u) = g(x,y)$$

① Construct $\dfrac{dy}{dx} = \dfrac{b \pm \sqrt{b^2 - ac}}{a}$, and solve this ODE.

② • if $b^2 - ac > 0$ then we get 2 equations, yielding two solutions ξ and η.

• if $b^2 - ac = 0$ then we get 1 equation for η. Then choose any ξ such that the mapping $(x,y) \to (\xi, \eta)$ is indeed a change of coordinates

• if $b^2 - ac < 0$ then we get two complex conjugate solutions, ϕ and ϕ^*. Then

$$\xi = \mathrm{Re}(\phi)$$

$$\eta = \mathrm{Im}(\phi)$$

③ Express the PDE in the new coordinate system.

Note: Be careful about $b(x,y)$ (the factor of 2)

$\Rightarrow$ If you are unsure, note that if the PDE is written as

$$\alpha(x,y)u_{xx} + \beta(x,y)u_{xy} + \gamma(x,y)u_{yy} + \mathcal{L}^{(1)}(u) = g(x,y)$$

then $\dfrac{dy}{dx} = \dfrac{\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$

and this is entirely equivalent to the previous case (since $\beta = 2b$, $\alpha = a$, $\gamma = c$).

$\boxed{3.1}$ $u_{xx} - 6u_{xy} + 9u_{yy} = xy^2$

(a) $\delta(\mathcal{L}) = 0 \Rightarrow$ a parabolic system.

To find the relevant change of variables $(x, y) \to (\xi, \eta)$, we need to solve

$$\eta_x - 3\eta_y = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -3 \Rightarrow y = -3x + \eta$$

To find ξ, we can take any function of (x, y) that always intercepts lines of constant $\eta = y + 3x$ $\Rightarrow$ take $\xi = x$

Then $\partial_x = \frac{\partial \eta}{\partial x}\frac{\partial}{\eta} + \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} = 3\frac{\partial}{\partial \eta} + \frac{\partial}{\partial \xi}$

$$\partial_y = \frac{\partial}{\partial \eta}$$

So note also that $u_{xx} - 6u_{xy} + 9u_{yy}$
$= (\partial_x - 3\partial_y)^2 u$

so

$$\partial_x - 3\partial_y = 3\partial_\eta + \partial_\xi - 3\partial_\eta = \partial_\xi$$

So the LHS of the operator is simply $\partial_{\xi\xi}$

and $\begin{cases} \eta = y + 3x \\ \xi = x \end{cases} \Rightarrow \begin{cases} y = \eta - 3\xi \\ x = \xi \end{cases}$

$\Rightarrow$ the canonical form of the equation is

$$u_{\xi\xi} = \xi(\eta - 3\xi)^2$$

This is not quite the required form:

To put it into the required form, simply set

$$t = \eta - 3\xi \quad \text{and} \quad \frac{\xi}{3} = s - t$$

So $\boxed{9\,u_{tt} = \frac{1}{3}(s-t)t^2}$ (works since $u_\xi = -3u_t \Rightarrow u_{\xi\xi} = 9u_{tt}$)

(b) To find the general solution, integrate wrt t:

$$u_{tt} = \frac{1}{27}(st^2 - t^3)$$

$$\Rightarrow u_t = \frac{1}{27}\left(s\frac{t^3}{3} - \frac{t^4}{4}\right) + h_1(s)$$

$$u = \frac{1}{27}\left(s\frac{t^4}{12} - \frac{t^5}{20}\right) + h_1(s)t + h_2(s)$$

$$\Rightarrow u(x,y) = \frac{1}{27}\left(\frac{(y+3x)y^4}{12} - \frac{y^5}{20}\right) + h_1(y+3x)\,y + h_2(y+3x)$$

Since $t = y$ and $s = y+3x$

(c) To fit to the bcs:

- $\sin x = h_2(3x) \Rightarrow h_2(x) = \sin\left(\frac{x}{3}\right)$
- $u_y(x,y) = \frac{1}{27}\left[\frac{y^3}{3}(y+3x) + \frac{y^4}{12} - \frac{y^4}{4}\right] + h_1(y+3x)$
  $+ y h_1'(y+3x) + h_2'(y+3x)$

so $\cos x = h_1(3x) + \frac{1}{3}\cos\left(\frac{x}{3}\right) \Rightarrow h_1(x) = \frac{2}{3}\cos\left(\frac{x}{3}\right)$

So finally $u(x,y) = \frac{1}{27}\left[\frac{(y+3x)y^4}{12} - \frac{y^5}{20}\right] + y\sin\left(\frac{y+3x}{3}\right) + \frac{2}{3}\cos\left(\frac{y+3x}{3}\right)$

$\boxed{3.2}$ $u_{xx} + 6u_{xy} - 16u_{yy} = 0$

(a) $\delta(\mathcal{L}) = 9 + 16 = 25 > 0 \Rightarrow$ hyperbolic

(b) To find the canonical form, we need to solve the two characteristics equations:

$$\xi_x + (3+5)\xi_y = 0 \qquad (1)$$
$$\eta_x + (3-5)\eta_y = 0 \qquad (2)$$

$(1) \Rightarrow \frac{dy}{dx} = 8 \Rightarrow y = 8x + \xi \qquad \xi = y - 8x$

$(2) \Rightarrow \frac{dy}{dx} = -2 \Rightarrow y = -2x + \eta \qquad \eta = y + 2x$

So
$$\partial_x = -8\partial_\xi + 2\partial_\eta$$
$$\partial_y = \partial_\xi + \partial_\eta$$
$$\partial_{xx} = (-8\partial_\xi + 2\partial_\eta)(-8\partial_\xi + 2\partial_\eta)$$
$$= 64\partial_{\xi\xi} - 32\partial_{\xi\eta} + 4\partial_{\eta\eta}$$
$$\partial_{yy} = \partial_{\xi\xi} + 2\partial_{\xi\eta} + \partial_{\eta\eta}$$
$$\partial_{xy} = (-8\partial_\xi + 2\partial_\eta)(\partial_\xi + \partial_\eta)$$
$$= -8\partial_{\xi\xi} - 6\partial_{\xi\eta} + 2\partial_{\eta\eta}$$

So $u_{xx} + 6u_{xy} - 16u_{yy} = \cancel{64u_{\xi\xi}} - 32u_{\eta\xi} + \cancel{4u_{\eta\eta}}$
$- \cancel{48u_{\xi\xi}} - 36u_{\eta\xi} + \cancel{12u_{\eta\eta}}$
$- \cancel{16u_{\xi\xi}} - 32u_{\eta\xi} - \cancel{16u_{\eta\eta}}$.

$\Rightarrow 100u_{\eta\xi} = 0$

(c) so $u = F(\eta) + G(\xi) = F(y+2x) + G(y-8x)$.

(d) To satisfy $u(-s, 2s) = s$

$u(s, 0) = \sin(2s)$

we repuire

$$s = F(2s - 2s) + G(2s + 8s)$$

$$\sin(2s) = F(0 + 2s) + G(0 - 8s)$$

the first epuation implies

$$G(10s) = s + \text{constant} = s - F(0)$$

$$\Rightarrow \quad G(s) = \frac{s}{10} - F(0)$$

the second implies

$$\sin(2s) = F(2s) + G(-8s)$$

$$= F(2s) - \frac{8s}{10} - F(0) = F(2s) - \frac{4s}{5} - F(0)$$

So $\quad F(s) = \sin(s) + \frac{2s}{5} + F(0)$

and finally

$$u(x,y) = \sin(y + 2x) + \frac{2}{5}(y + 2x) + \frac{(y - 8x)}{10}$$

$\boxed{3.3}$ $u_{xx} + 4u_{xy} + u_x = 0$

(a) $\delta(\mathcal{L}) = 4 \Rightarrow$ hyperbolic

To find the two sets of characteristics we must solve

$$\xi_x + (2+2)\xi_y = 0 \qquad (1)$$

$$\eta_x + (2-2)\eta_y = 0 \qquad (2)$$

(1) $\Rightarrow$ $\dfrac{dy}{dx} = 4$ so $y = 4x + \xi \Rightarrow \xi = y - 4x$

(2) $\Rightarrow$ $\dfrac{\partial \eta}{\partial x} = 0$ so η is a function of y only $\rightarrow$ choose $\eta = y$.

So $\partial_x = \dfrac{\partial \xi}{\partial x}\partial_\xi + \dfrac{\partial \eta}{\partial x}\partial_\eta = -4\partial_\xi$

$\partial_y = \dfrac{\partial \xi}{\partial y}\partial_\xi + \dfrac{\partial \eta}{\partial y}\partial_\eta = \partial_\xi + \partial_\eta$

$\Rightarrow$ $\partial_{xx} = 16\partial_{\xi\xi}$

$\partial_{xy} = -4\partial_\xi(\partial_\xi + \partial_\eta) = -4\partial_{\xi\xi} - 4\partial_{\eta\xi}$

$u_{xx} + 4u_{xy} + u_x = 16u_{\xi\xi} - 16u_{\xi\xi} - 16u_{\eta\xi} - 4u_\xi = 0$

$\Rightarrow$ $\boxed{u_{\eta\xi} = -\frac{1}{4}u_\xi}$

(b) Integrate wrt ξ

$u_\xi = e^{-\frac{1}{4}\eta} \cdot f(\xi)$ arbitrary function of η

so $$u = e^{-\frac{1}{4}\eta} F(\xi) + G(\eta)$$

Check:

$$u_\xi = e^{\frac{1}{4}\eta} \frac{dF}{d\xi}$$

$$u_{\eta\xi} = -\frac{1}{4} e^{\frac{1}{4}\eta} \frac{dF}{d\xi} = -\frac{1}{4} u_\xi \quad \square$$

(c) To satisfy the initial conditions

$$u(s, 8s) = 0$$

$$u_x(s, 8s) = 4e^{-2s}$$

- first express u in terms of (x, y)

$$u(x,y) = e^{-\frac{1}{4}y} F(y - 4x) + G(y)$$

$$\Rightarrow \quad u_x(x,y) = -4e^{-\frac{1}{4}y} F'(y-4x)$$

Then set

$$0 = e^{-\frac{1}{4}\cdot 8s} F(8s - 4s) + G(8s) \qquad ①$$

$$4e^{-2s} = -4e^{-\frac{1}{4}8s} F'(8s - 4s) \qquad ②$$

so ②: $F'(4s) = -1 \quad \Rightarrow \quad F(s) = -s + F(0)$

① $e^{-2s} F(4s) + G(8s) = 0$

$$\Rightarrow e^{-2s}\left(-4s + F(0)\right) + G(8s) = 0$$

$$\Rightarrow \quad G(s) = \left[\frac{s}{2} - F(0)\right] e^{-\frac{s}{4}}$$

so

$$u(x,y) = e^{-\frac{1}{4}y}\left(4x - y + F(0)\right) + \left[\frac{y}{2} - F(0)\right] e^{-y/4}$$

$$= e^{-y/4}\left\{4x - y + \frac{y}{2}\right\} = e^{-y/4}\left\{4x - \frac{y}{2}\right\}$$