2.2.4 Method of characteristics for quasilinear equations

General form: in (x,y) space

QL equations can be written as

$$a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u)$$

- The equations determining the characteristics are similar to the semilinear case:

$$\begin{cases} \dfrac{\partial x}{\partial \tau} = a(x,y,u) \\ \dfrac{\partial y}{\partial \tau} = b(x,y,u) \\ \dfrac{\partial u}{\partial \tau} = c(x,y,u) \end{cases}$$

such that

$$\left(\frac{\partial x}{\partial \tau}\right)_s \frac{\partial u}{\partial x} + \left(\frac{\partial y}{\partial \tau}\right)_s \frac{\partial u}{\partial y} = \left(\frac{\partial u}{\partial \tau}\right)_s$$

- Note, however, that now the equations for $x^{(s)}(\tau)$ and $y^{(s)}(\tau)$ depend on the <u>value</u> of the function u itself so the system of ODEs is fully coupled.

Recall:

<u>Semilinear case</u>

$$\begin{cases} \dfrac{\partial x}{\partial \tau} = a(x,y) \\ \dfrac{\partial y}{\partial \tau} = b(x,y) \end{cases}$$

$$\frac{\partial u}{\partial \tau} = c(x,y,u)$$

$\Rightarrow$ The characteristics are independent of the value of the function u, and notably independent of $u_0(s)$ (of the initial condition)

The system decouples.

This time, the characteristics depend on the initial condition $u_0(s)$ of the system.

Definition:

- The characteristic curves are the 3D solutions of the system

$$\begin{cases} \frac{\partial x}{\partial \tau} = a(x, y, u) \\ \frac{\partial y}{\partial \tau} = b(x,y,u) \\ \frac{\partial u}{\partial \tau} = c(x,y,u) \end{cases}$$

and are parametrized as $\mathcal{C}^{(s)} = \begin{pmatrix} x^{(s)} \\ y^{(s)} \\ u^{(s)} \end{pmatrix}$

- The characteristics are the projection of the characteristic curves onto the (x,y) plane. They are parametrized as $\gamma^{(s)} = \begin{pmatrix} x^{(s)} \\ y^{(s)} \end{pmatrix}$

- For semilinear problems, characteristics can be calculated first, while $u^{(s)}(\tau)$ is calculated later to determine the solution

- In quasilinear problems, the characteristics cannot be calculated directly $\rightarrow$ the system is solved for the characteristic curves. $\begin{pmatrix} x^{(s)}(\tau) \\ y^{(s)}(\tau) \\ u^{(s)}(\tau) \end{pmatrix}$

The method is otherwise similar.

Example 1

$$\begin{cases} x\, u_x - u\, u_y = y \\ u(1,y) = y \end{cases}$$

① Initial condition curve

Let $\begin{cases} x_0(s) = 1 \\ y_0(s) = s \\ u_0(s) = s \end{cases}$ then $u(x_0(s), y_0(s)) = u(1,s) = s$

② Characteristic curves:

$$\begin{cases} \frac{\partial x}{\partial \tau} = x \\ \frac{\partial y}{\partial \tau} = -u \\ \frac{\partial u}{\partial \tau} = y \end{cases}$$

$\Rightarrow \quad x = x_0(s) e^{\tau}$

a system of two coupled ODEs. Combine these to get

$$\frac{\partial^2 y}{\partial \tau^2} = -\frac{\partial u}{\partial \tau} = -y$$

So $\begin{cases} y = A \sin\tau + B\cos\tau \\ u = -\frac{\partial y}{\partial \tau} = -A\cos\tau + B\sin\tau \end{cases}$

Apply initial conditions

$$\begin{cases} x = e^{\tau} \\ y = -s\sin\tau + s\cos\tau = s(\cos\tau - \sin\tau) \\ u = s\cos\tau + s\sin\tau = s(\cos\tau + \sin\tau) \end{cases}$$

So $\tau = \ln x$ and $s = \dfrac{y}{\cos\tau - \sin\tau} = \dfrac{y}{\cos(\ln x) - \sin(\ln x)}$

So $u = \dfrac{y\,(\cos(\ln x) + \sin(\ln x))}{\cos(\ln x) - \sin(\ln x)}$

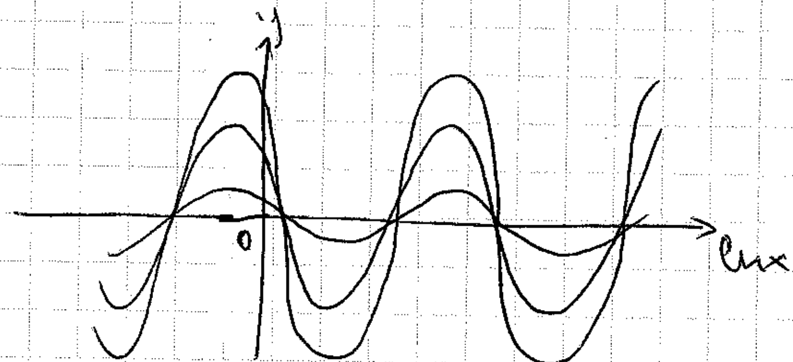
$$u(x,y) = y\,\frac{1 + \tan(\ln x)}{1 - \tan(\ln x)}$$

Question: ① What do the characteristics look like in (x,y) plan

② Where is the solution defined?

① $y = s\left(\cos(\ln x) - \sin(\ln x)\right)$

So naturally y is an oscillatory function of $\ln x$ with amplitude ranging from $-s\sqrt{2}$ to $+s\sqrt{2}$



zeros are at
$\ln x = \frac{\pi}{4} + k\pi$
$(x = e^{\frac{\pi}{4}+k\pi}$

→ Naturally, all characteristics cross at points $\begin{cases} x = e^{\frac{\pi}{4}+k\pi} \\ y = 0 \end{cases}$

② When characteristics cross, the system

$\begin{cases} x(s,\tau) \\ y(s,\tau) \end{cases}$ is not invertible into $\begin{cases} s(x,y) \\ \tau(x,y) \end{cases}$

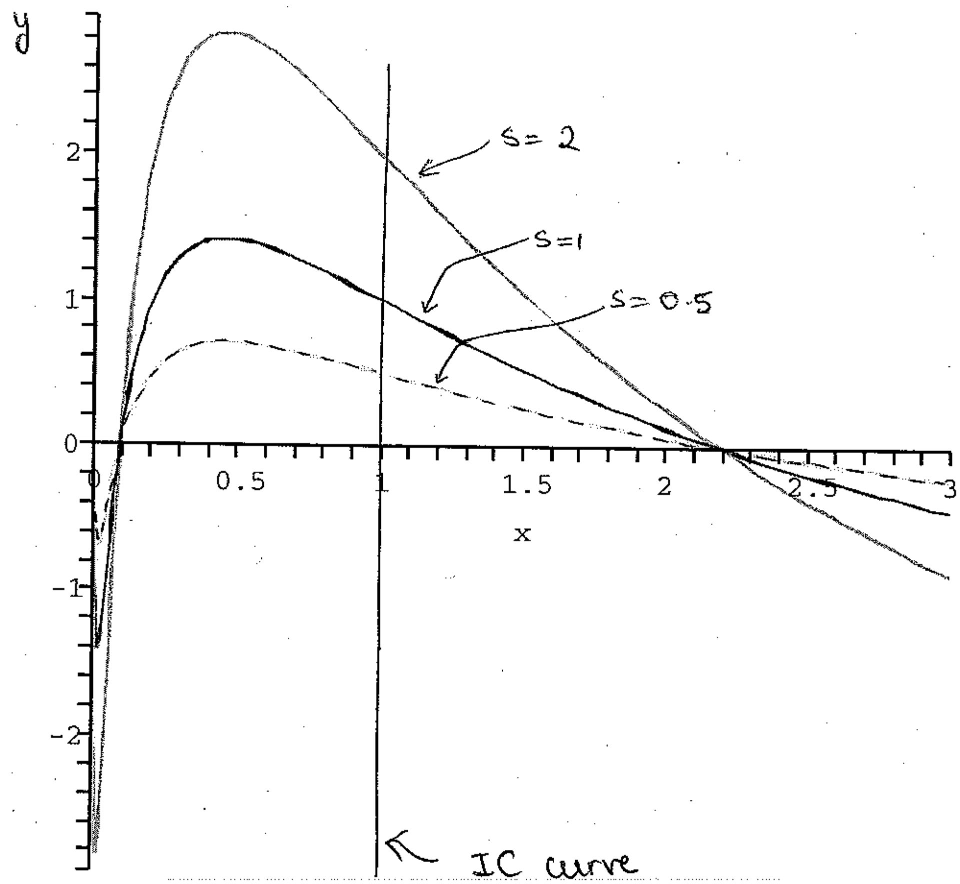
→ the solution is defined for
$$e^{-\frac{\pi}{4}} < x < e^{\pi/4}$$
but not outside of that interval

This corresponds to $u(x,y) = y\,\dfrac{1+\tan(\ln x)}{1-\tan(\ln x)}$

with the requirement
$$\tan(\ln x) \neq 1 .$$

Characteristics of the system

$$\begin{cases} x u_x - u u_y = y \\ u(1, y) = y \end{cases}$$



Example 2. $\left\{ \begin{array}{l} \text{Same PDE with} \\ u(1, y) = -y \end{array} \right.$

→ initial condition is slightly different. (same position on the $(x-y)$ plane, but a different value for u)

$$\begin{cases} x_0(s) = 1 \\ y_0(s) = s \\ u_0(s) = -s \end{cases}$$

→ Only difference is that

$$\begin{cases} x = e^{\tau} \\ y = s\,(\sin\tau + \cos\tau) \\ u = s\,(\sin\tau - \cos\tau) \end{cases}$$

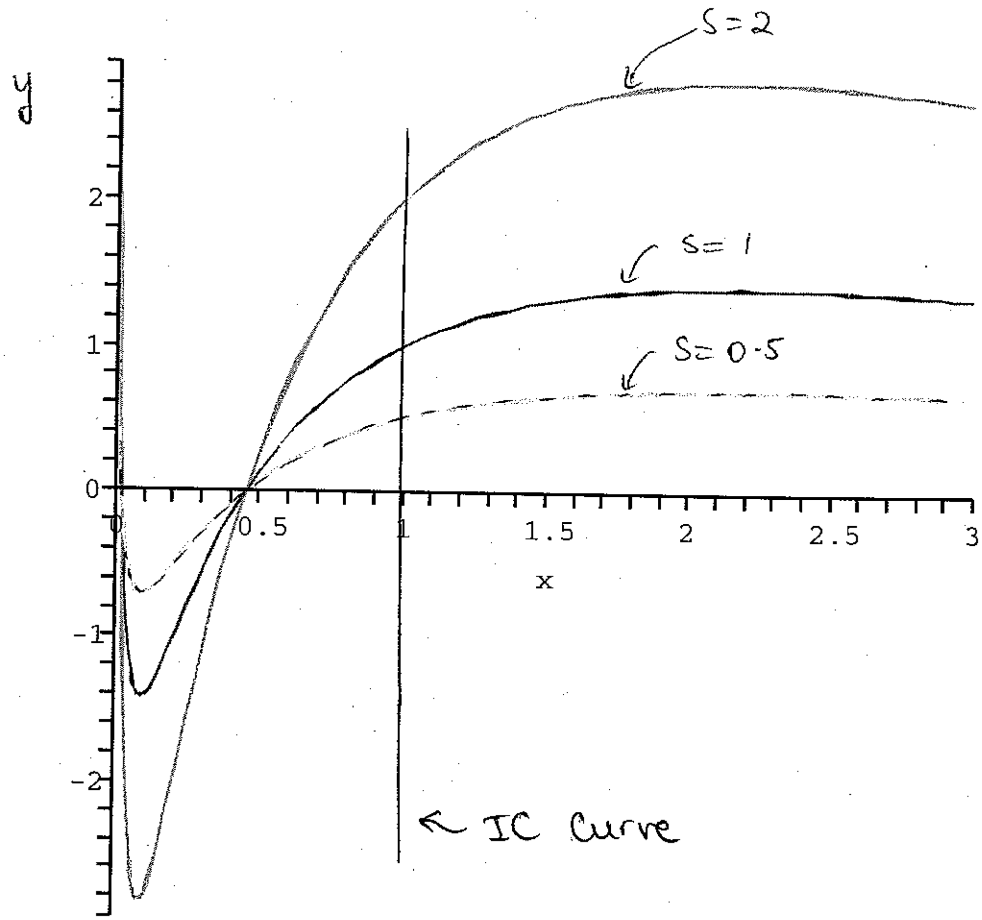
so the characteristics are now

$$y = s\left(\sin(\ln x) + \cos(\ln x)\right) \Rightarrow \text{different from previous case}$$

This is a specific property of quasilinear equations vs semilinear equations: the characteristics are not uniquely defined by the PDE but also by the initial conditions. This effect is a consequence of the nonlinearity of the problem.

Characteristics of the system

$$\begin{cases} x u_x - u u_y = y \\ u(1,y) = -y \end{cases}$$



2.3 Existence and uniqueness

2.3.1 Introduction

- We are finding that the existence of a solution is associated with the invertibality of the mapping between the (s,τ) space and the (x,y) space.
- In some examples (see previously), this implied that the solution was only defined in a subset of $\mathbb{R}^2$
- Can worse situations happen? Yes!

Let's compare two examples

PDE 1: $\quad x u_x + (x+y) u_y = u+1$

PDE 2: $\quad x u_x + y u_y = u+1$

with initial condition

$$u(x,x) = x^2 \quad : \quad \begin{cases} x_0 = s \\ y_0 = s \\ u_0 = s^2 \end{cases}$$

Case 1: Integrate

$$\begin{cases} \dfrac{\partial x}{\partial \tau} = x & \Rightarrow \quad x = x_0(s) e^{\tau} \\ \dfrac{\partial y}{\partial \tau} = x+y & \Rightarrow \quad \dfrac{\partial y}{\partial \tau} = x_0(s) e^{\tau} + y \\ \dfrac{\partial u}{\partial \tau} = u+1 & \Rightarrow \quad u = \left(u_0(s)+1\right) e^{\tau} - 1 \end{cases}$$

To solve for y, use an integrating factor method (for example)

$$\frac{dy}{d\tau} - y = x_0(s) e^{\tau}$$

so $\mu = e^{-\tau}$ and

$$\frac{d}{d\tau}\left(ye^{-\tau}\right) = x_0(s)$$

$$\rightarrow \quad ye^{-\tau} = C + x_0(s)\tau$$

so $y = Ce^{\tau} + x_0(s)\tau e^{\tau}$

To ensure $y = y_0(s)$ when $\tau = 0$ choose

$$y = y_0(s)e^{\tau} + x_0(s)\tau e^{\tau}$$

So finally $\begin{cases} x = se^{\tau} \\ y = se^{\tau}(\tau+1) \\ u = (s^2+1)e^{\tau} - 1 \end{cases}$

so $\frac{y}{x} = \tau + 1 \Rightarrow \tau = \frac{y}{x} - 1$

so $s = xe^{-\tau} = xe^{-\left(\frac{y}{x}-1\right)}$

and therefore

$$u = \left[x^2 e^{-2\left(\frac{y}{x}-1\right)} + 1\right] e^{\frac{y}{x}-1} - 1$$

$\Rightarrow$ provided $x \neq 0$ no problem he

Case 2: The characteristics are obtained by integrating

$$\begin{cases} \frac{\partial x}{\partial \tau} = x \rightarrow x = x_0(s)e^{\tau} = se^{\tau} \\ \frac{\partial y}{\partial \tau} = y \rightarrow y = y_0(s)e^{\tau} = se^{\tau} \\ \frac{\partial u}{\partial \tau} = u+1 \rightarrow u = \left[s^2+1\right]e^{\tau} - 1 \end{cases}$$

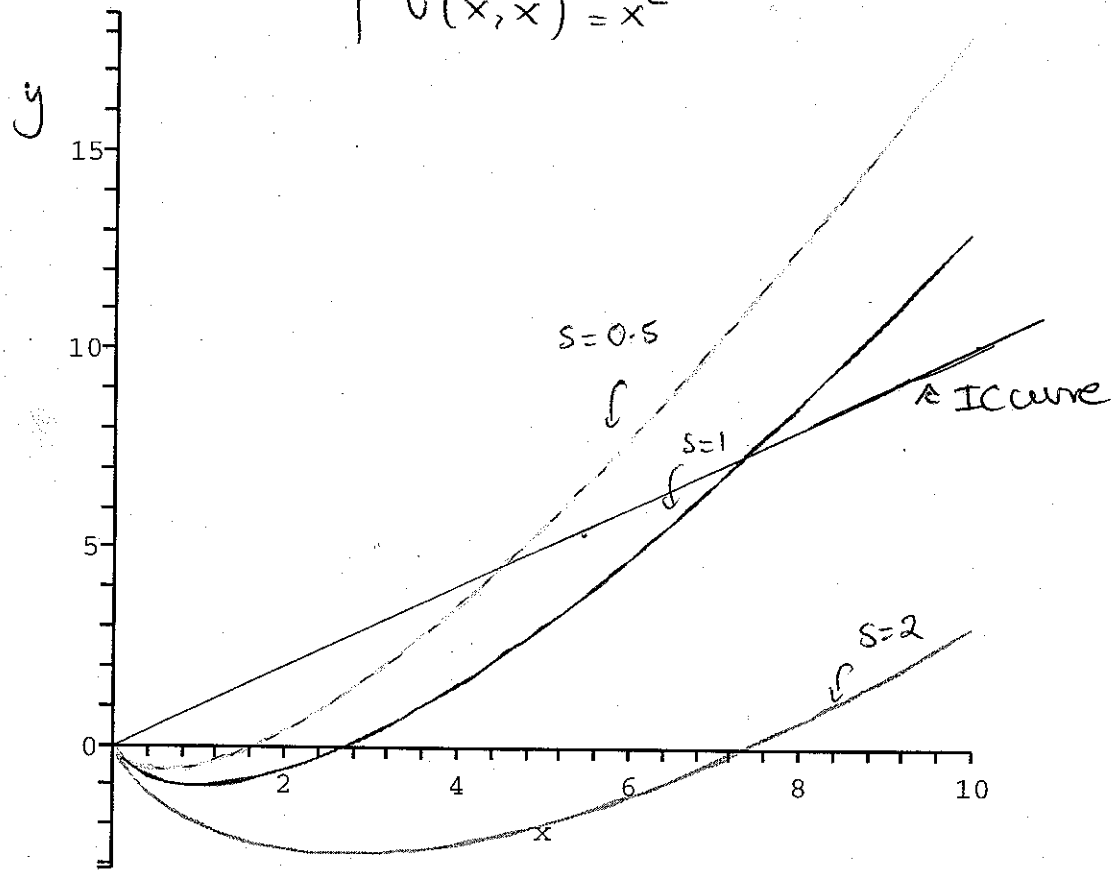
Similarly, we try to invert the mapping we find

$$\frac{y}{x} = 1$$

$\rightarrow$ the mapping $\begin{cases} x = se^{\tau} \\ y = se^{\tau} \end{cases}$ only maps the $x = y$ line.

Characteristics for the system

$$\begin{cases} x u_x + (x+y) u_y = u + 1 \\ u(x,x) = x^2 \end{cases}$$



$\Rightarrow$ $u(s,\tau)$ exists for all s and τ but we cannot invert the mapping for x and y

Moreover, the initial condition $u(x,x) = x^2$ does not satisfy the PDE on Γ:

$$xu_x + yu_y$$
$$= x(2x) + x(2x) = 4x^2 \neq x^2$$

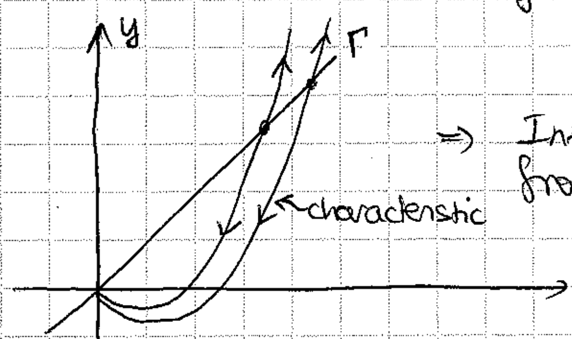
$\rightarrow$ No solutions to this problem.

What is $\neq$ between these two cases in terms of the characteristics of the system?

Case 1 The characteristics are given by $x = s e^{\frac{y}{x}+1}$ or equivalently

$$y = x\left[\ln\left(\frac{x}{s}\right) - 1\right]$$

$\rightarrow$

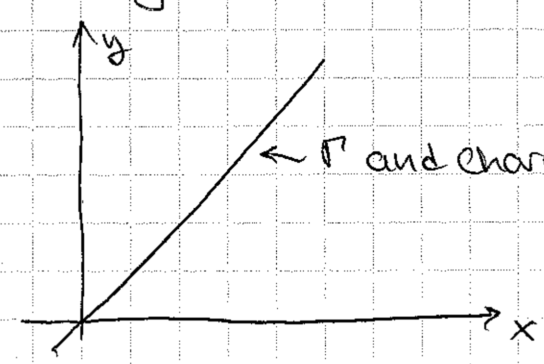


$\Rightarrow$ Information is transported away from Γ on characteristics.

$\Rightarrow$ No problem until the characteristic reaches $x = 0$; there, the mapping is not invertible

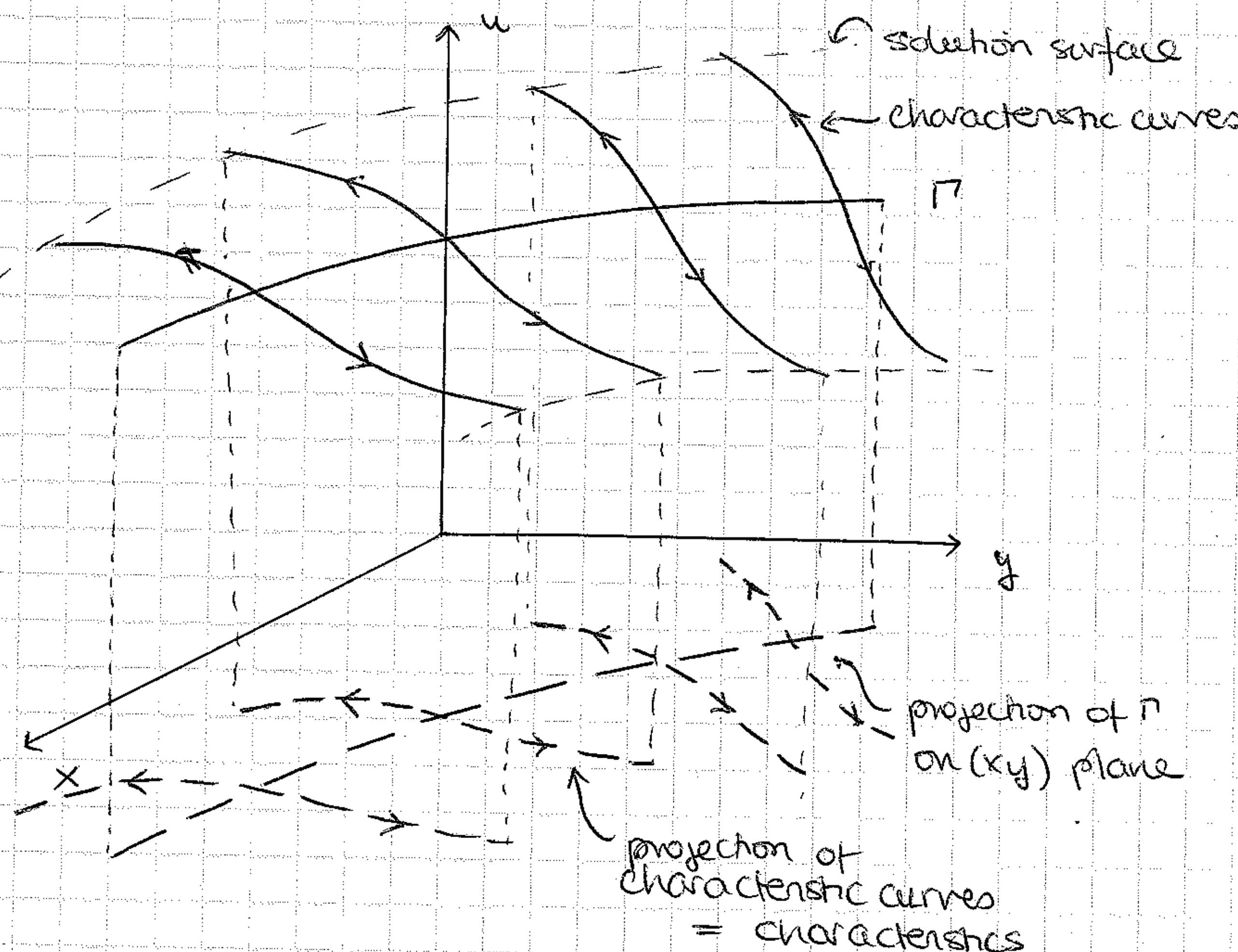
Case 2 The characteristics are $y = x$ which is also the equation for the initial condition curve.

$\Rightarrow$ the initial information cannot be transported away from Γ.



2.3.2 Existence and uniqueness theorem

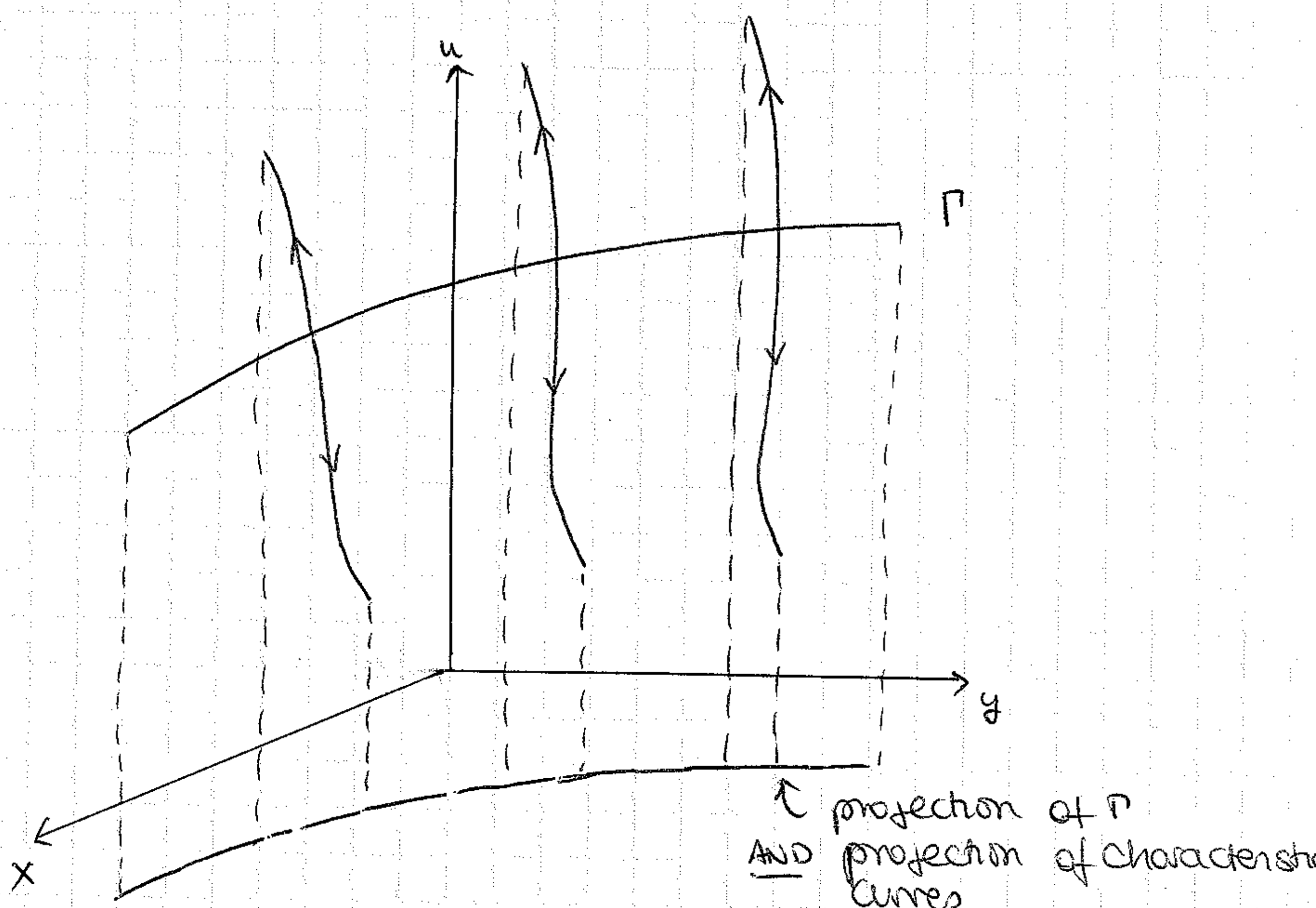
- In a given well-posed problem (solution exists and is unique), consider the surface defined as the set of points $(x, y, u(x,y))$ in the (x, y, u) space
- This surface contains Γ (initial condition curve)
- This surface is spanned by the characteristic curves. The solution $u(x,y)$ is propagated along the characteristic curves away from Γ.



- The projection of Γ and of the characteristic curves on the $(x-y)$ plane shows the characteristics intersecting the projection of Γ
  $\rightarrow$ solution is indeed propagated away from Γ

• In an ill-posed problem, two situations may arise

<u>Case 1</u>: no solutions to the PDE



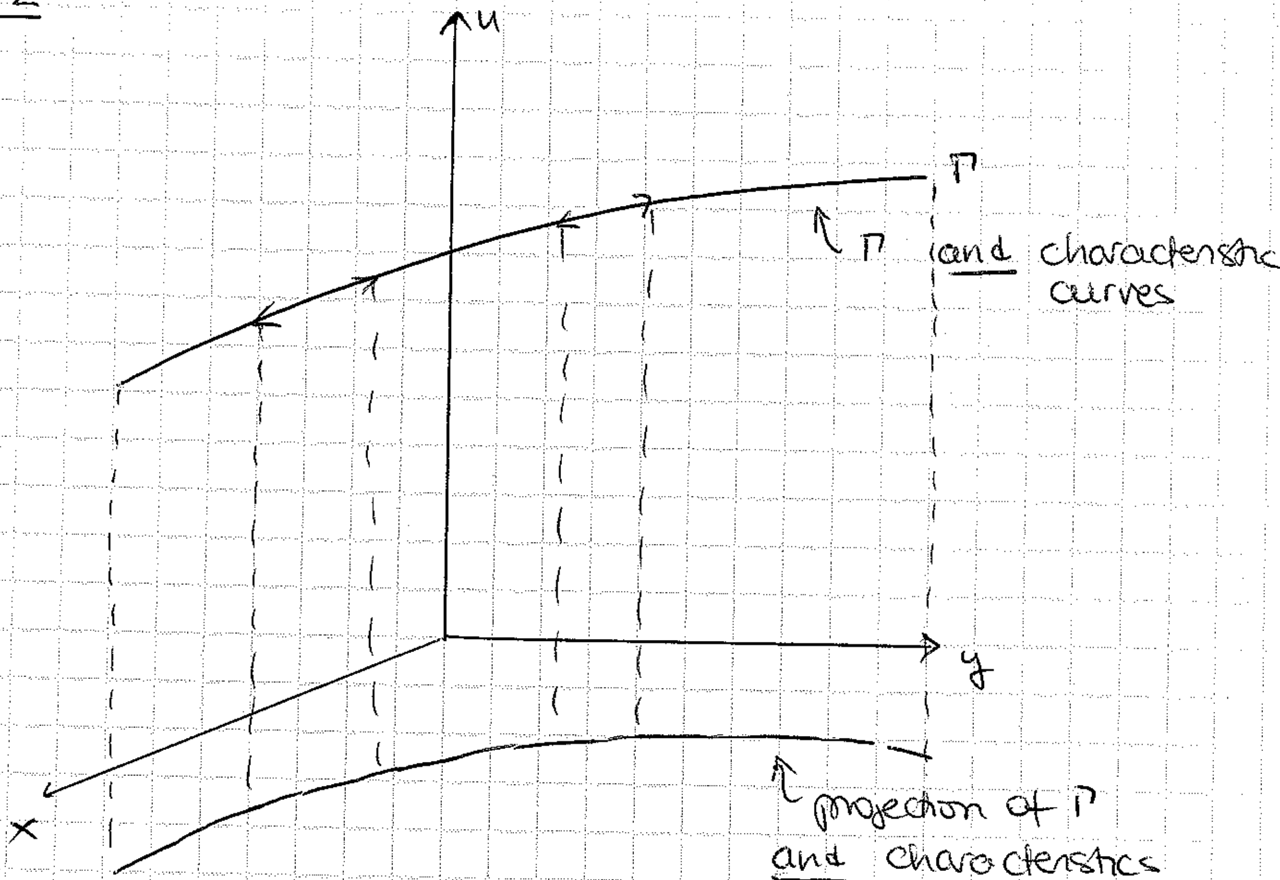
In this case: characteristic curves intercept the Γ curve, but the surface spanned by Γ and the characteristic curves entirely projects on a single curve in the (x,y) plane

⇒ the solution is <u>not</u> propagated away from Γ in a well-defined way

→ for every pt (x,y) on the projection of Γ ∃ an infinite # of values of u coming from each characteristic curve

⇒ here there are <u>no solutions to the PDE</u>

Case 2



⇒ This time the only constraint on the solution is that the surface $(x, y, u(x,y))$ must pass through Γ. Although the characteristic curves do not propagate the solution away from Γ, they do lie on Γ.

⇒ any surface which passes through Γ is a solution of the PDE

⇒ there are an ∞ # of solutions to the PDE.

⇒ The difference between the well-posed case and the ill-posed cases is clearly seen in the projection of Γ and the projection of the characteristic curves (the characteristics)

- If the characteristics intercept the projection of Γ ⇒ well posed problem

- If the characteristics are // to the projection of Γ
  $\Rightarrow$ ill posed problem.

Mathematically

Two vectors in the x-y plane intersect (i.e are not //) provided they have non-zero cross product

At a point s on the initial curve, the tangent vector is

$$\begin{pmatrix} dx_0/ds \\ dy_0/ds \\ du_0/ds \end{pmatrix}$$

$\Rightarrow$ its projection on (x-y) is $\begin{pmatrix} dx_0/ds \\ dy_0/ds \\ 0 \end{pmatrix}$

The characteristic curve emanating from s has tangent vector

$$\begin{pmatrix} dx/d\tau \\ dy/d\tau \\ du/d\tau \end{pmatrix}_{(x_0,y_0,u_0)} = \begin{pmatrix} a(x_0,y_0,u_0) \\ b(x_0,y_0,u_0) \\ c(x_0,y_0,u_0) \end{pmatrix} \rightarrow \text{its projection is } \begin{pmatrix} a(x_0,y_0,u_0) \\ b(x_0,y_0,u_0) \\ 0 \end{pmatrix}$$

The transversality condition @ a point s is therefore satisfied provided

$$\begin{pmatrix} dx_0/ds \\ dy_0/ds \\ 0 \end{pmatrix} \times \begin{pmatrix} a(x_0,y_0,u_0) \\ b(x_0,y_0,u_0) \\ 0 \end{pmatrix} \neq 0$$

$$\Leftrightarrow \quad b(x_0,y_0,u_0)\frac{dx_0}{ds} - a(x_0,y_0,u_0)\frac{dy_0}{ds} \neq 0$$

Theorem: • Assume that $a(x,y,u)$, $b(x,y,u)$ and $c(x,y,u)$ are smooth functions in a neighborhood of the initial curve (x_0, y_0, u_0)
• Assume that the transversality condition holds for each $s \in [s_0 - 2\delta, s_0 + 2\delta]$ on the initial curve

then: $\exists$ a unique solution $u(x,y)$ in the neighborhood of the initial curve defined by $\tau \in [-\epsilon, \epsilon]$, $s \in [s_0-\delta, s_0+\delta]$

Idea behind the proof (see Pinchover & Rubinstein for detail)

- given a system of ODEs for the characteristic curves

$$\begin{cases} \dfrac{\partial x}{\partial \tau} = a(x,y,u) \\ \dfrac{\partial y}{\partial \tau} = b(x,y,u) \\ \dfrac{\partial u}{\partial \tau} = c(x,y,u) \end{cases}$$

we can always find a solution that satisfies the initial conditions

$$\begin{cases} x(\tau=0) = x_0(s_0) \\ y(\tau=0) = y_0(s_0) \\ u(\tau=0) = u_0(s_0) \end{cases}$$ from a point s_0 on the initial curve

in a neighborhood of $\tau=0$ (properties of dynamical systems) provided a, b & c are smooth functions near (x_0, y_0, u_0).

$\Rightarrow$ We can always find $\begin{cases} x(\tau,s) \\ y(\tau,s) \\ u(\tau,s) \end{cases}$ in a neighborhood of $\tau=0$, $s=s_0$

provided the initial condition curve is continuous near s_0.

- The problem of existence and uniqueness lies in the inversion of the system to obtain $u(x,y)$

let's write $x(\tau,s) = x(0,s_0) + \tau\left(\dfrac{\partial x}{\partial \tau}\right)_{\substack{\tau=0\\ s=s_0}} + (s-s_0)\left(\dfrac{\partial x}{\partial s}\right)_{\substack{\tau=0\\ s=s_0}}$

$$y(\tau,s) = y(0,s_0) + \tau\left(\frac{\partial y}{\partial \tau}\right)_{\substack{\tau=0\\ s=s_0}} + (s-s_0)\left(\frac{\partial y}{\partial s}\right)_{\substack{\tau=0\\ s=s_0}}$$

This is also

$$x = \underbrace{x_0(s_0)}_{\text{from initial conditions}} + \tau\,\underbrace{a(x_0,y_0,u_0)}_{\text{from PDE \& characteristic equation}} + (s-s_0)\underbrace{\left(\frac{\partial x_0}{\partial s}\right)_{s=s_0}}_{\text{from initial condition}}$$

and $y = y_0(s_0) + \tau\, b(x_0,y_0,u_0) + (s-s_0)\left(\dfrac{\partial y_0}{\partial s}\right)_{s=s_0}$

Now to invert these equations to obtain τ and s in terms of x and y we have the matrix equation

$$\begin{pmatrix} a(x_0, y_0, u_0) & \left.\frac{\partial x_0}{\partial s}\right)_{s_0} \\ b(x_0, y_0, u_0) & \left.\frac{\partial y_0}{\partial s}\right)_{s_0} \end{pmatrix} \begin{pmatrix} \tau \\ s \end{pmatrix} = \begin{pmatrix} x - x_0(s_0) + s_0 \frac{\partial x_0}{\partial s} \\ y - y_0(s_0) + s_0 \frac{\partial y_0}{\partial s} \end{pmatrix}$$

$\Rightarrow$ this system has a unique solution provided

$$\begin{vmatrix} a(x_0, y_0, u_0) & \frac{\partial x_0}{\partial s} \\ b(x_0, y_0, u_0) & \frac{\partial y_0}{\partial s} \end{vmatrix} \neq 0$$

<u>As required</u>.

<u>Example</u> Given the PDE

$$x u u_x + y u u_y = u^2 - 1$$

with the initial condition $u(x, x^2) = x^3$ for $x \in [a, b]$

for what values of (a, b) will there be a unique solution?

- initial condition curve $x_0(s) = s$
  $y_0(s) = s^2$
  $u_0(s) = s^3$

$$a(x_0, y_0, u_0) = x_0 u_0 = s^4$$
$$b(x_0, y_0, u_0) = y_0 u_0 = s^5$$
$$\frac{\partial x_0}{\partial s} = 1 \qquad \frac{\partial y_0}{\partial s} = 2s$$

$$\Rightarrow \begin{vmatrix} a & \frac{\partial x_0}{\partial s} \\ b & \frac{\partial y_0}{\partial s} \end{vmatrix} = \begin{vmatrix} s^4 & 1 \\ s^5 & 2s \end{vmatrix} = 2s^5 - s^5 = s^5$$

$\Rightarrow$ as long as $s \neq 0$ then $\exists$ a unique solution. So any interval <u>excluding</u> $s=0$ will lead to a unique solution

Exercise: find the solution for $(a, b) = (0, +\infty)$.
(be careful with absolute values!)