CHAPTER 2 First order PDEs (in 2 dimensions)

2.1 General formulae

A first order PDE in 2 dimensions is in the form of

$$\mathcal{F}\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right) = 0$$

- A first order linear PDE in 2 dimensions is

$$a(x,t)\frac{\partial u}{\partial t} + b(x,t)\frac{\partial u}{\partial x} = c(x,t)u + d(x,t).$$

NONLINEAR PDES:

- A first order semilinear PDE in 2D is

$$a(x,t)\frac{\partial u}{\partial t} + b(x,t)\frac{\partial u}{\partial x} = c(x,t,u)$$

- A first order quasilinear PDE is

$$a(x,t,u)\frac{\partial u}{\partial t} + b(x,t,u)\frac{\partial u}{\partial x} = c(x,t,u).$$

A fully nonlinear first order PDE is none of the above!

2.2 Method of characteristics for quasilinear equations

2.2.1 Warmup example

Let's study $u_t = c_0 u + c_1(x,t)$ c_0 constant

Note that for each x, it is actually an ODE in t
$\rightarrow$ fix x and solve it!

Use integrating factor method (for example)

$$u_t - c_0 u = c_1(x,t)$$

$\Rightarrow$ We try to find an integrating factor $\mu(x,t)$ such that

$$\mu u_t - \mu C_0 u = \mu G(x,t)$$

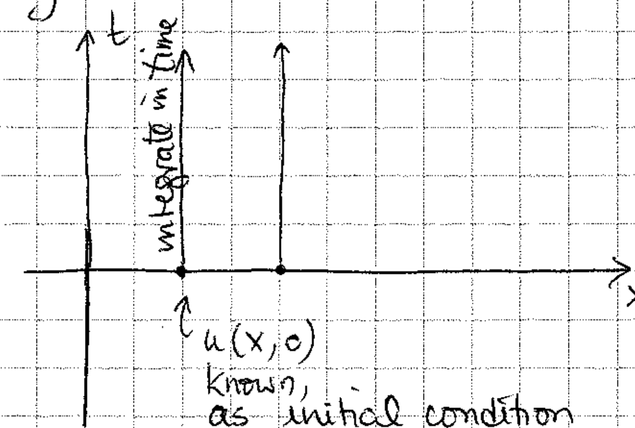
$$= \frac{\partial}{\partial t}(\mu u)$$

$\longrightarrow$ take $\mu = e^{-C_0 t}$ so

$$\frac{\partial}{\partial t}\left(e^{-C_0 t} u\right) = e^{-C_0 t} G(x,t)$$

$$e^{-C_0 t} u(x,t) - e^{-C_0 \cdot 0} u(x,0) = \int_{t'=0}^{t'=t} e^{-C_0 t'} G(x,t')\,dt'$$

Again, this can be done for each value of x separately: we are solving the equation by integrating along lines of constant x



<u>Initial conditions</u> (u is known at $t=0$)

Suppose we require that $u(x,0) = 3x$ then

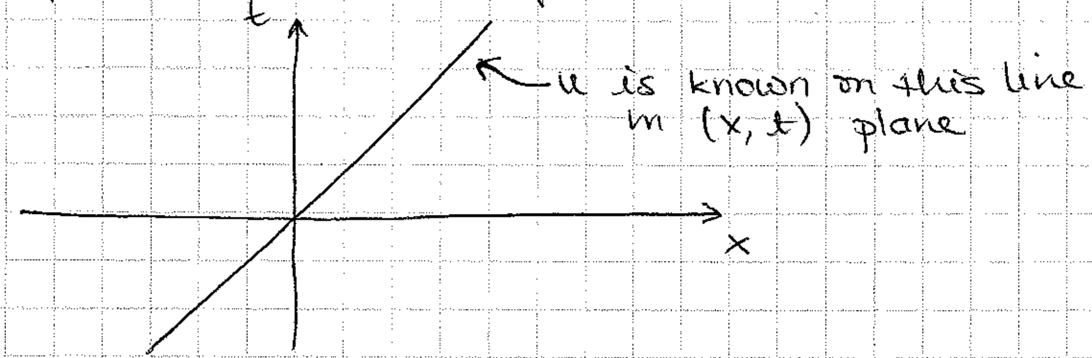
$$u(x,t) = e^{+C_0 t} u(x,0) + \int_{t'=0}^{t'=t} e^{-C_0(t'-t)} G(x,t')\,dt'$$

$$= 3x e^{+C_0 t} + \int_{t'=0}^{t'=t} e^{-C_0(t'-t)} G(x,t')\,dt'$$

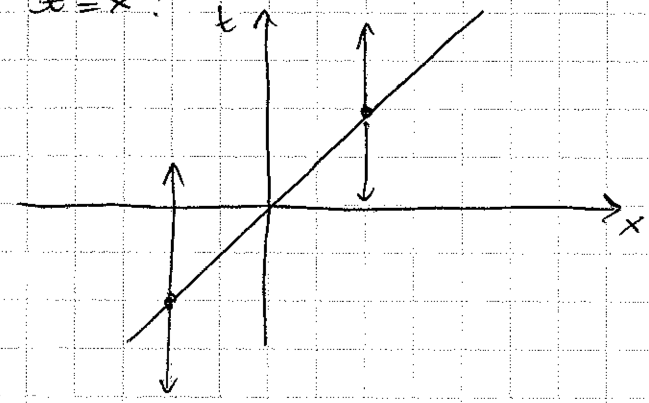
$\longrightarrow$ a unique solution.

Other kinds of additional condition

① Suppose instead we require that $u(x,x) = 3x$



Then, instead of integrating from $t'=0$, we integrate from $t'=x$:



Mathematically:

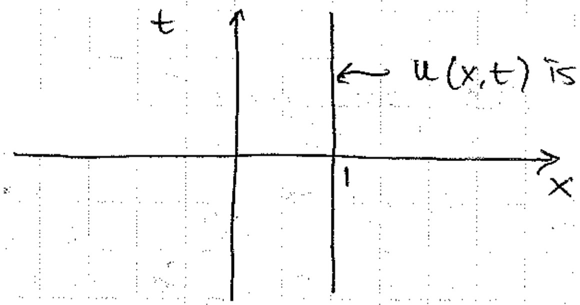
$$e^{-c_0 t} u(x,t) - e^{-c_0 x} u(x,x) = \int_{t'=x}^{t'=t} e^{-c_0 t'} G(x,t')\, dt'$$

$$\Rightarrow \quad e^{-c_0 t} u(x,t) = e^{-c_0 x} \cdot 3x + \int_x^t e^{-c_0 t'} G(x,t')\, dt'$$

$$u(x,t) = e^{-c_0 (x-t)} \cdot 3x + \int_x^t e^{-c_0 (t'-t)} G(x,t')\, dt'$$

→ again, there is a unique solution to the PDE with the given additional condition.

(2) Now suppose we set $C_0 = 0$ and try to impose as additional condition $u(1,t) = 2t$



<u>Problem!</u> The additional condition doesn't satisfy the equation

$$\frac{\partial u}{\partial t} = 2 \quad \rightarrow \quad u_t - C_0 u = 2 - 2C_0 t \neq 0 .$$

$\rightarrow$ there are no solutions to the equation!

(3) Now suppose $u(1,t) = 2e^{C_0 t}$ then

$$u_t - C_0 u = 2C_0 e^{C_0 t} - 2C_0 e^{C_0 t} = 0 \quad \checkmark$$

$\Rightarrow$ the additional condition satisfies the equation

But note that any function of the form

$$u(x,t) = f(x)\, e^{C_0 t}$$

satisfies the PDE <u>and</u> the additional condition provided $f(1) = 2$

$\Rightarrow$ there are an ∞ of solutions to the problem!

<u>Conclusion</u>: • Depending on the additional conditions chosen, there can be one, no or an ∞ of solutions to the problem. Case (1) is <u>well-posed</u> while cases (2) and (3) are <u>ill-posed</u>

• What is the difference between cases (1), (2) and (3)?

Note that in case (1), the additional condition crosses all lines of constant x, while in cases (2) and (3), the additional condition <u>is</u> a line of constant x.

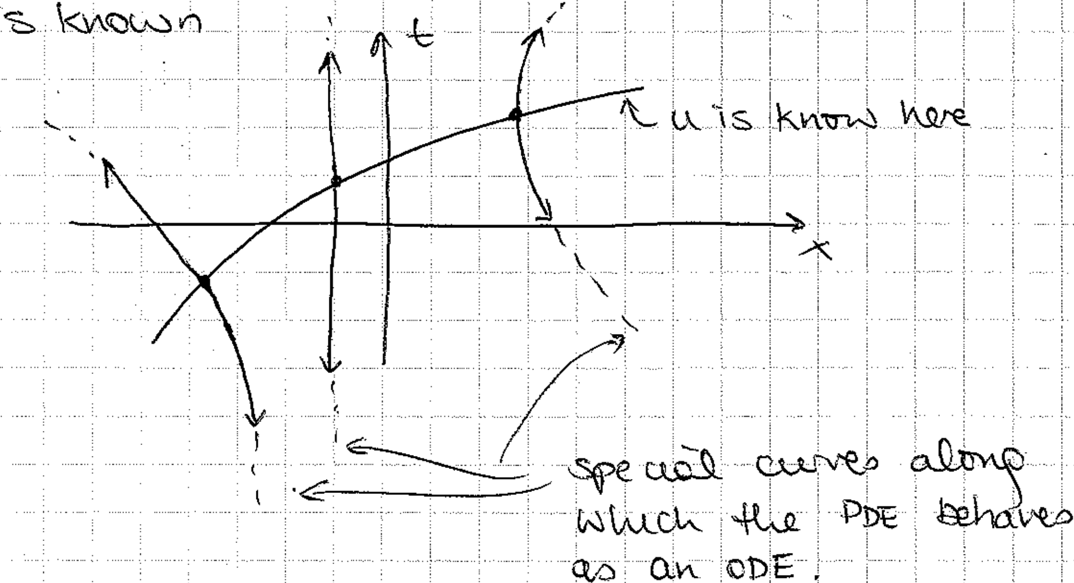
2.2.2 Gearing up to the general method

Now consider the linear transport equation with constant coefficients.

$$a \frac{\partial u}{\partial t} + b \frac{\partial u}{\partial x} = c_1 u + c_0$$

where a, b, c_1, c_0 are constants.

<u>Idea</u>: we would like to find curves (as before) along which we could integrate the PDE <u>as if</u> it were an ODE, from an initial or additional condition line where $u(x,t)$ is known



DETOUR: Review of parametric curves

Any curve in $\mathbb{R}^n$ can be represented by a set of parametric equations

$$\begin{cases} x_1 = f_1(s) \\ x_2 = f_2(s) \\ \vdots \\ x_n = f_n(s) \end{cases}$$

where s is the parameter.

<u>Examples</u>: A circle in $\mathbb{R}^2$ (x,y) centred on $(0,0)$ has the equation $\begin{cases} x = R\cos(s) \\ y = R\sin(s) \end{cases}$ where R is the radius

- A straight line in $\mathbb{R}^2$ has the parametric equation

$$\begin{cases} x = as + c \\ y = bs + d \end{cases}$$

<u>Check</u>: eliminate s to get

$$y = b\left(\frac{x-c}{a}\right) + d = \frac{b}{a}x + \left(d - \frac{bc}{a}\right)$$

<u>Property of parametric curves</u>

The tangent vector to the curve $\{f_1(s) \dots f_n(s)\}$ is

$$\underline{df} = \begin{pmatrix} df_1/ds \\ df_2/ds \\ \vdots \\ df_n/ds \end{pmatrix}$$

<u>Examples</u>: • the tangent vector to the line

$$\begin{cases} x = as + c \\ y = bs + d \end{cases} \quad \text{is} \quad \underline{df} = \begin{pmatrix} dx/ds \\ dy/ds \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

• Suppose you are travelling from SC to Big Sur. Your <u>trajectory</u> is given by the parametric curve

$$\begin{pmatrix} x(t) \\ y(t) \\ h(t) \end{pmatrix} \begin{matrix} \leftarrow \text{latitudinal position } x \\ \leftarrow \text{longitudinal position } y \\ \leftarrow \text{height} \end{matrix}$$

Your <u>velocity</u> is the tangent vector to the trajectory

$$\underline{v} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dh}{dt} \end{pmatrix} \begin{matrix} \leftarrow \text{North-South velocity} \\ \leftarrow \text{East-West velocity} \\ \leftarrow \text{vertical velocity.} \end{matrix}$$

<u>Note</u>: A parametrization is <u>NOT</u> unique.

<u>Example</u>: $\begin{cases} x = R\sin s \\ y = R\cos s \end{cases}$ and $\begin{cases} x = R\sin(s^2) \\ y = R\cos(s^2) \end{cases}$ represent the <u>same</u> curve

Back to the first order PDE

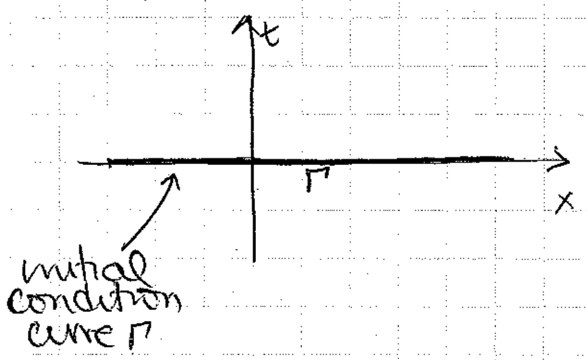
Step 1: We represent the additional condition curve as a parametric curve with parameter s

Suppose we know $u(x,t)$ on a particular curve Γ in the (x,t) plane. Let's parametrize Γ with the functions $x_0(s)$, $t_0(s)$ such that

$$\Gamma = \begin{cases} x_0(s) \\ t_0(s) \end{cases}$$

then on this curve $u(x_0(s), t_0(s)) = u_0(s)$

Examples: • Suppose we want to impose $u(x, 0) = 3x$

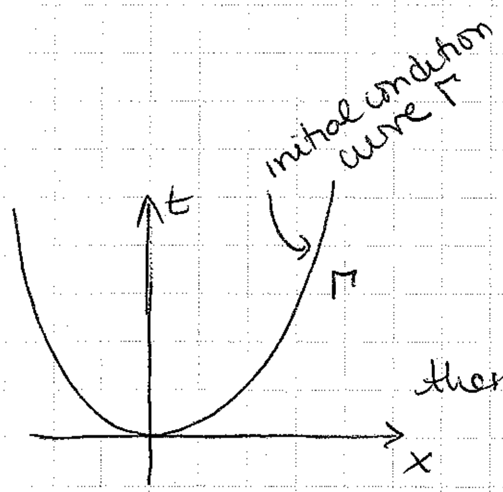


The initial condition curve has $t=0$ for all x $\Rightarrow$ Parametrize it (for example) as

$$\begin{cases} x_0(s) = s \\ t_0(s) = 0 \end{cases} \Rightarrow u_0(s) = 3s$$

or we could also use

$$\begin{cases} x_0(s) = s^2 \\ t_0(s) = 0 \end{cases} \Rightarrow u_0(s) = 3s^2$$



• Suppose $u(x, x^2) = e^x$

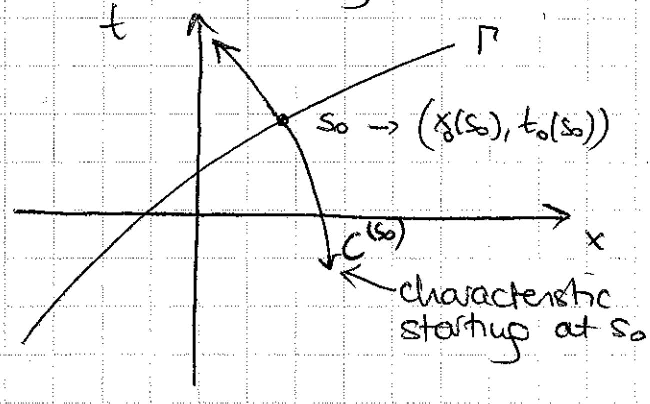
then
$$\begin{cases} x_0(s) = s \\ t_0(s) = s^2 \end{cases} \Rightarrow u_0(s) = e^s$$

Note: Since there are many possible parametric representations of the same curve, always try to choose the simplest one:

Prefer
$$\begin{cases} x_0(s) = s \\ t_0(s) = s^2 \end{cases}$$
over
$$\begin{cases} x_0(s) = \ln(s^2) \\ t_0(s) = [\ln(s^2)]^2 \end{cases}$$
!

<u>Step 2</u>: Now that we have parametrized the "initial" condition curve we want to identify special curves along which the PDE behaves as an ODE. These are called characteristics.

① Suppose that for a selected point on the initial curve there exists only <u>one</u> characteristic emanating from it



⟹ let's parametrize this characteristic with the new parameter τ

$$C^{(s_0)} : \begin{cases} x^{(s_0)}(\tau) \\ t^{(s_0)}(\tau) \end{cases}$$

⟹ on this curve

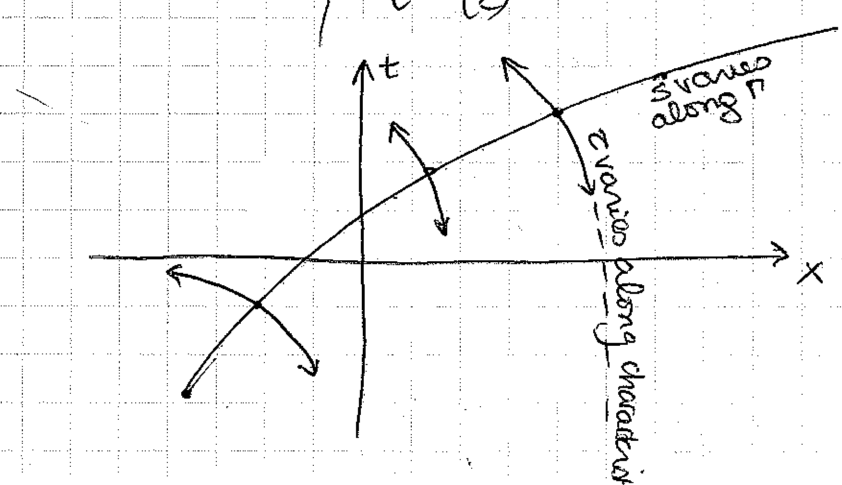
$$u\left(x^{(s_0)}(\tau), t^{(s_0)}(\tau)\right) = u^{(s_0)}(\tau)$$

u depends on τ only

② Now let's do this same construction for every point $[x_0(s), t_0(s)]$ on the initial condition curve

⟹ We get a family of characteristics, each starting from a point identified with the parameter s, and each parametrized with τ:

$$C^{(s)} \begin{cases} x^{(s)}(\tau) \\ t^{(s)}(\tau) \end{cases} \quad \text{with } u^{(s)}(\tau) = u\left(x^{(s)}(\tau), t^{(s)}(\tau)\right)$$



$\Rightarrow$ Note that what we have really done, is to remap the (x,t) space onto the (s,τ) space

So that the function

$$u(x,t) \text{ is also } u\left(x^{(s)}(\tau), t^{(s)}(\tau)\right)$$
$$= u(s,\tau)$$

with the added requirement that the PDE

$a\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x}$ behaves like an ODE

when restricted to a characteristic $(s = \text{constant})$

How do we impose this requirement?

Note that $\left.\frac{\partial u}{\partial \tau}\right)_s$ is the derivative of u along a characteristic. (i.e holding s constant)

derivative of u w.r.t parameter τ at constant s

By multivariate chain rule, and using $\begin{pmatrix} x^{(s)}(\tau) \\ t^{(s)}(\tau) \end{pmatrix}$ on characte

$$\left.\frac{\partial u}{\partial \tau}\right)_s = \frac{\partial u}{\partial x}\left.\frac{\partial x}{\partial \tau}\right)_s + \frac{\partial u}{\partial t}\left.\frac{\partial t}{\partial \tau}\right)_s$$
$$= \frac{\partial u}{\partial x}\frac{d}{d\tau}\left[x^{(s)}\right] + \frac{\partial u}{\partial t}\frac{d}{d\tau}\left[t^{(s)}\right]$$

Going back to the original PDE, if

$$a\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} = \frac{d}{d\tau}\left[t^{(s)}\right]\frac{\partial u}{\partial t} + \frac{d}{d\tau}\left[x^{(s)}\right]\frac{\partial u}{\partial x}$$

then $\Rightarrow \left.\frac{\partial u}{\partial \tau}\right)_s = c_1 u + c_0$

Now the PDE looks like an ODE for s held constant, i.e, along a characteristic.

This occurs when $\boxed{\frac{dt^{(s)}}{d\tau} = a \qquad \frac{dx^{(s)}}{d\tau} = b}$

$$\Rightarrow \quad C^{(s)} = \begin{cases} t^{(s)} = a\tau + \text{constant specific to this characteristic} \\ x^{(s)} = b\tau + \text{constant} \quad " \quad " \quad " \quad " \end{cases}$$

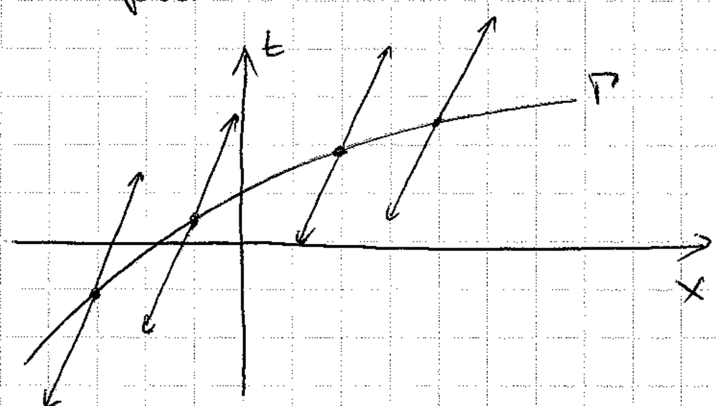
Suppose we require that when $\tau = 0$ we are on the initial condition curve then at $\tau = 0$

$$t^{(s)} = t_0(s)$$
$$x^{(s)} = x_0(s)$$

thus $C^{(s)} = \begin{cases} t^{(s)} = a\tau + t_0(s) \\ x^{(s)} = b\tau + x_0(s) \end{cases}$ is the parametric equation for the characteristic emanating from the point $[x_0(s), t_0(s)]$ on the initial condition curve.

<u>What do these characteristics look like?</u>

Here, they are straight lines with slope $\frac{a}{b}$ in the (x,t) plane



Note: this is only true when the PDE has constant coefficients (see later)

<u>Step 3</u> <u>What is the solution to the PDE?</u>

Now we have to solve $\frac{\partial u}{\partial \tau} = c_1 u + c_0$ for each s

$$\Rightarrow \quad u = A(s)e^{c_1\tau} - \frac{c_0}{c_1} + \frac{c_0}{c_1}e^{c_1\tau} \quad \text{(check this)}$$

where the arbitrary constant $A(s)$ is chosen such that $u = u_0(s)$ when $\tau = 0$:

$$u(s,\tau) = u_0(s)e^{c_1\tau} - \frac{c_0}{c_1} + \frac{c_0}{c_1}e^{c_1\tau}$$

How do we get a solution in terms of (x,t) ?

Invert the system (if possible)

$$\begin{cases} t = a\tau + t_0(s) \\ x = b\tau + x_0(s) \end{cases}$$

to write τ and s in terms of x and t, then plug into $u(s,\tau)$

From here on it is easier to look at more specific example

Example 1 Suppose we want to solve the simple transport equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + V_0 \dfrac{\partial u}{\partial x} = 0 \\ u(x,0) = e^{-x^2/2} \end{cases} \qquad \text{(a Gaussian)}$$

Step 1 : Parametrize the initial condition

$$\begin{cases} x_0(s) = s \\ t_0(s) = 0 \\ u_0(s) = e^{-s^2/2} \end{cases}$$

Step 2 The characteristic curves are such that

$$\begin{pmatrix} a = 1 \\ b = V_0 \end{pmatrix} \qquad \begin{cases} \dfrac{\partial t^{(s)}}{\partial \tau} = 1 \\ \dfrac{\partial x^{(s)}}{\partial \tau} = V_0 \end{cases} \Rightarrow \begin{cases} t^{(s)} = \tau + t_0(s) \\ x^{(s)} = V_0\tau + x_0(s) \end{cases} \Rightarrow \begin{cases} t^{(s)} = \tau \\ x^{(s)} = V_0\tau + s \end{cases}$$

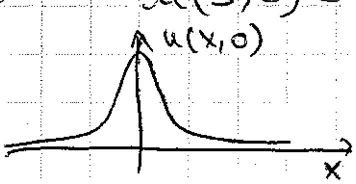
Step 3 : The solution to $\dfrac{\partial u}{\partial \tau} = 0$ $\quad (c_1 = c_0 = 0)$

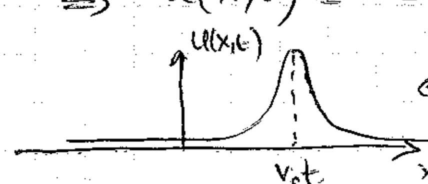
is $\quad u =$ constant on a characteristic

$$\Rightarrow \quad u = u_0(s) = e^{-s^2/2}$$

Step 4 : $$\begin{cases} t = \tau \\ x = V_0\tau + s \end{cases} \Rightarrow \begin{cases} \tau = t \\ s = x - V_0 t \end{cases}$$

so $$u(s,\tau) = e^{-s^2/2} \Rightarrow u(x,t) = e^{-(x - V_0 t)^2/2}$$





← a "travelling" Gaussian i.e horizontally moved.

Step 5   Always check your answer

$$\frac{\partial u}{\partial t} = -V_0 \, (x - V_0 t) \, e^{-(x-V_0 t)^2/2}$$

$$\frac{\partial u}{\partial x} = (x - V_0 t) \, e^{-\frac{(x-V_0 t)^2}{2}}$$

$$\frac{\partial u}{\partial t} + V_0 \frac{\partial u}{\partial x} = 0 \quad \checkmark$$

Example 2 : General case

$$\begin{cases} \dfrac{\partial u}{\partial t} + V_0 \dfrac{\partial u}{\partial x} = 0 \\ u(x,0) = F(x) \end{cases}$$

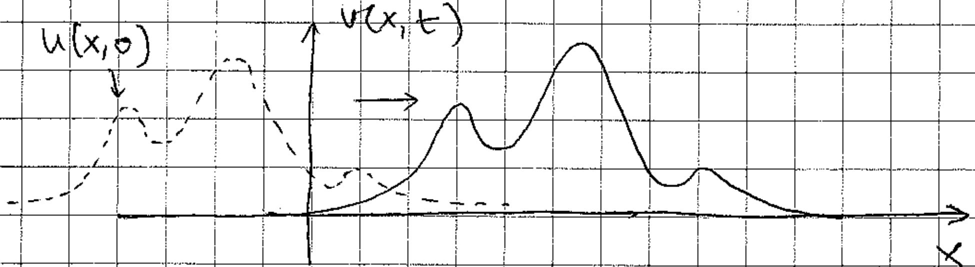
Step 1 :
$$\begin{cases} x_0(s) = s \\ t_0(s) = 0 \\ u_0(s) = F(s) \end{cases}$$

Step 2 :
$$\begin{cases} \dfrac{dt}{d\tau} = 1 \\ \dfrac{dx}{d\tau} = V_0 \end{cases} \Rightarrow \begin{cases} t = \tau \\ x = V_0 \tau + s \end{cases}$$

Step 3
$$\frac{du}{d\tau} = 0 \Rightarrow u = u_0(s) = F(s)$$

Step 4 :
$$\begin{cases} \tau = t \\ s = x - V_0 t \end{cases} \rightarrow \boxed{u(x,t) = F(x - V_0 t)}$$

$\Rightarrow$ The initial condition $F(x)$ "moves" with velocity V_0



Example 3 :

$$\begin{cases} \dfrac{\partial u}{\partial t} + x\dfrac{\partial u}{\partial x} = 0 \\ u(x,0) = e^{-x^2} \end{cases}$$

Step 1 :

$$\begin{cases} x_0(s) = s \\ t_0(s) = 0 \\ u_0(s) = e^{-s^2} \end{cases}$$

Step 2

$$\begin{cases} \dfrac{dt}{d\tau} = 1 \\ \dfrac{dx}{d\tau} = x \end{cases} \Rightarrow \begin{cases} t = \tau + t_0(s) = \tau \\ x = x_0(s)e^{\tau} = se^{\tau} \end{cases}$$

Step 3

$$\frac{du}{d\tau} = 0 \Rightarrow u = u_0(s) = e^{-s^2}$$

Step 4 .

$$\begin{cases} \tau = t \\ s = xe^{-t} \end{cases} \Rightarrow u(\cancel{x}, t) = e^{-(xe^{-t})^2} = e^{-\frac{x^2}{e^{2t}}}$$

This describes a Gaussian with constant amplitude 1 constant mean, but with a width which grows exponentially in time.

2.3.3 Semilinear equations

The method for semilinear equations

$$\begin{cases} a(x,t)\dfrac{\partial u}{\partial t} + b(x,t)\dfrac{\partial u}{\partial x} = c(x,t,u) \\ u(x,0) = \phi(x) \end{cases}$$

is the same as for linear equations. However, note that the resulting ODE for u will be nonlinear.

Example

$$\begin{cases} \dfrac{\partial u}{\partial t} + x\dfrac{\partial u}{\partial x} = -e^{-u} \\ u(x,0) = e^{-x^2} \end{cases}$$

Step 1

$$\Gamma \begin{cases} x_0(s) = s \\ t_0(s) = 0 \\ u_0(s) = e^{-s^2} \end{cases}$$

Step 2

$$\begin{cases} \dfrac{dt}{d\tau} = 1 \\ \dfrac{dx}{d\tau} = x \end{cases} \Rightarrow \begin{cases} t = \tau + t_0(s) = \tau \\ x = x_0(s)e^{\tau} = se^{\tau} \end{cases}$$

Step 3

$$\frac{du}{d\tau} = -e^{-u} \Rightarrow \frac{du}{e^{-u}} = -d\tau \Rightarrow e^{u}\,du = -d\tau$$

$$e^{u} = -\tau + K(s) \Rightarrow u = \ln(\tau + K(s))$$

$$\text{At} \quad \tau = 0 \quad e^{u_0(s)} = K(s) \Rightarrow K(s) = e^{e^{-s^2}}$$

$$u = \ln\left(-\tau + e^{e^{-s^2}}\right)$$

Step 4

$$\begin{cases} \tau = t \\ s = xe^{-t} \end{cases} \Rightarrow u(x,t) = \ln\left(-t + e^{e^{-x^2e^{-2t}}}\right)$$

Step 5 Check:

$$\frac{\partial u}{\partial t} = \frac{-1 + 2x^2e^{-2t}\,e^{-x^2e^{-2t}}\,e^{e^{-x^2e^{-2t}}}}{-t + e^{e^{-x^2e^{2t}}}}$$

$$\frac{\partial u}{\partial x} = -\frac{2xe^{-2t}\,e^{-x^2e^{-2t}}\,e^{e^{-x^2e^{-2t}}}}{-t + e^{e^{-x^2e^{-2t}}}}$$

so ✓.

What did we learn?

① Method of solution of semilinear, first order PDES

Step 1: Parametrize the initial condition curve

Step 2: If $a(x,t)\frac{\partial u}{\partial t} + b(x,t)\frac{\partial u}{\partial x} = c(x,t,u)$

then the characteristics are found by solving the system

$$\left\{\begin{array}{l} \frac{\partial t^{(s)}}{\partial \tau} = a(x,t) \\ \\ \frac{\partial x^{(s)}}{\partial \tau} = b(x,t) \end{array}\right\} \text{Note that these are \underline{coupled} ODEs.}$$

with the initial condition

$$t^{(s)}(\tau=0) = t_0(s)$$
$$x^{(s)}(\tau=0) = x_0(s)$$

Step 3: The solution to the PDE in (s,τ) is found by solving

$$\frac{\partial u^{(s)}}{\partial \tau} = c(x,t,u)$$

(note that x and t depend on s and τ)

Step 4: If possible, invert the system

$$\begin{cases} t(s,\tau) \\ x(s,\tau) \end{cases} \text{ to get } \begin{cases} \tau(x,t) \\ s(x,t) \end{cases}$$

and plug into $u(s,\tau)$ to get $u(x,t)$.

Step 5 Check answer.

② Note :

- When the linear PDE is homogeneous ($c(x,t) = 0$) then

$$\frac{\partial u}{\partial \tau} = 0 \Rightarrow$$ u is constant along characteristics, in other words, the characteristics are contour levels of the solution $u(x,t)$.

- When the PDE is not homogeneous then u is not constant along characteristics. The characteristics propagate the initial condition according to the equation

$$\frac{\partial u^{(s)}}{\partial \tau} = c\left(x^{(s)}(\tau), t^{(s)}(\tau), u^{(s)}(\tau)\right)$$

(see examples later)

③ Question

- What is the condition for the mapping $\begin{cases} x(s,\tau) \\ t(s,\tau) \end{cases}$ to be invertible?

- What happens if the characteristics are somewhere parallel to the initial condition curve?