(II) Green's functions in higher dimensions; the Poisson equation

In what follows, we now move to the problem of Green's functions in 2 or more dimensions, with specific application to the Poisson equation.

$$\begin{cases} \nabla^2 u = f(\underline{r}) & r \in D \\ \text{Some boundary condition} & r \in \partial D . \end{cases}$$

As we shall see, for this particular kind of problem the existence and uniqueness of solutions is not guaranteed.

(1) Existence & uniqueness of solutions

The problem with the Poisson equation comes from the following property:

$$\int_V \nabla \cdot u \, dV = \int_{\partial V} \underline{u} \cdot \underline{n} \, ds \qquad \text{the divergence theorem}$$

$\underline{n}$ = unit vector normal to surface (outward).

Since, by definition, $\nabla^2 u = \nabla \cdot (\nabla u)$ then the integral of the Poisson equation over the domain D is

$$\int_D \nabla \cdot (\nabla u) \, dV = \underbrace{\int_{\partial V} \underline{\nabla u} \cdot \underline{n} \, ds}_{\substack{\text{depends on} \\ \text{boundary} \\ \text{conditions} \\ \text{(sometimes known)}}} = \underbrace{\int_D f(\underline{r}) \, dV}_{\substack{\text{depends on} \\ \text{forcing} \\ \text{(known)}}} .$$

$\Rightarrow$ The nature of the equation itself provides a constraint on the "flux" of $\underline{u}$ through the boundary and the integrated source term.

The mathematical constraint can be interpreted physically once more by thinking of the Poisson equation as the steady-state version of a heat diffusion process $\Rightarrow$

in equilibrium the heat flux through the boundary of a domain must be equal to the integrated heat sources in the domain.

However, we see here that this is a very general constraint.

!! In particular, it implies that the Poisson equation with Neumann-type boundary conditions (i.e when

$\underline{\nabla u} \cdot \underline{n}$ is known on the boundary)

does NOT have a solution unless $\int_{\partial D} \nabla u \cdot n \, ds = \int_{D} f(r)\, dV$

In general: • the problem of existence of solutions for elliptic equations is much more complex than for parabolic / hyperbolic equations.

• Provided the domain considered is bounded and smooth enough.

- Solutions to the Dirichlet problem exist and are unique
- solutions to the Neumann problem exist (if $*$ holds) but are not unique ($u = v + K$, $K \in \mathbb{R}$ is also solution)
- solutions to the Robin problem exist and are unique.

See Pinchover & Rubinstein textbook, for good proofs + much more.

As in 1D, the solution to $\nabla^2 u = f(\underline{r})$ with homogeneous boundary conditions can be written as

$$u(\underline{r}) = \int_D G(\underline{r}, \underline{r}')\, f(\underline{r}')\, d^n\underline{r}'$$

Volume integral in n dimension

where the Green's function G is the solution of

$$\nabla^2 G(\underline{r}, \underline{r}') = \delta(\underline{r} - \underline{r}').$$

with the same boundary conditions.

The problem of "finding" G, however, is often quite difficult.

Of course if separation of variables can be done, then the Green's function can be constructed by eigenfunction expansion (see previous Chapters for examples). In other cases, additional methods exist.

② The Green's functions for the unbounded domain

The solution to $\nabla^2 G(\underline{r}, \underline{r}') = \delta(\underline{r} - \underline{r}')$ in the infinite domain is easily found from the solution of

$$\nabla^2 G(\underline{r}) = \delta(\underline{r}) \quad \text{by translation symmetry}$$

Meanwhile, the solution to $\nabla^2 G = \delta(\underline{r})$ is point symmetric, and so becomes an equation in r only when expressed in a polar coordinate system (in 2D) or a spherical coordinate system (in 3D).

<u>Example in 2D</u> : the infinite domain Green's function is s.t

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) = \delta(r)$$

→ this is really $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) = 0$ everywhere but $r=0$

⇒ we seek a solution singular at $r=0$:

$$\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) = 0$$

$$\Rightarrow \quad r\frac{\partial G}{\partial r} = K \quad \Rightarrow \quad \frac{\partial G}{\partial r} = \frac{K}{r}$$

$$\Rightarrow \quad G(r) = K' + K\ln r$$

- Typically, BCs in an ∞ domain require $G \to 0$ as $r \to \infty$. Here of course, it's a bit of a problem... Let's take $K' = 0$. (sweep under carpet).
- The remaining constant comes from the requirement that the integral of a δ-function over any domain be one ⇒ on a "unit circle" for example

$$\int_{\substack{\text{unit}\\\text{circle}}} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) \cdot r\,dr\,d\theta = \int_{\substack{\text{unit}\\\text{circle}}} \delta(r)\, r\,dr\,d\theta = 1$$

$$\Rightarrow \int_0^1 \frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) 2\pi\, dr = 1$$

$$\Rightarrow \quad 2\pi\left[r\frac{\partial G}{\partial r}\right]_0^1 = 1$$

$$\Rightarrow \quad \left.\frac{\partial G}{\partial r}\right|_{r=1} = \frac{1}{2\pi} \quad \Rightarrow \quad K = \frac{1}{2\pi}$$

$$\Rightarrow \quad \boxed{K = \frac{1}{2\pi}}$$

$$\Rightarrow \quad G(r) = \frac{1}{2\pi}\ln r$$

$$\Rightarrow \quad \boxed{G(r, r') = \frac{1}{2\pi}\ln(|r - r'|)}$$

Example of application:

The steady state temperature distribution on an infinite conducting plate subject to heating $H(x,y)$ is the solution of

$$\nabla^2 T = -H(x,y)$$

and so $$T(x,y) = \int_{\mathbb{R}^2} -H(x',y')\, G(x,x';y,y')\, dx'dy'$$

$$T(x,y) = \iint_{\mathbb{R}^2} -H(x',y')\, \ln\left((x-x')^2+(y-y')^2\right) \frac{dx'dy'}{4\pi}.$$

whatever the function H is.

Example in 3D. The infinite domain Green's function is solution of $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial G}{\partial r}\right) = \delta(r)$, i.e, the singular solution of $$\frac{\partial}{\partial r}\left(r^2\frac{\partial G}{\partial r}\right) = 0.$$

$$\Rightarrow \quad r^2 \frac{\partial G}{\partial r} = k \quad \Rightarrow \quad \frac{\partial G}{\partial r} = \frac{k}{r^2}$$

$$\Rightarrow \quad G = k' - \frac{k}{r}$$

Again, if we require $G \to 0$ as $r \to \infty$, we can take $k' = 0$

The integral of $\delta(\underline{r})$ over a unit sphere must be one hence

$$\int_{\substack{\text{unit}\\ \text{sphere}}} \cancel{\frac{1}{r^2}} \frac{\partial}{\partial r}\left(r^2 \frac{\partial G}{\partial r}\right) \cancel{r^2}\, dr \sin\theta\, d\theta\, d\phi = 1$$

$$\Rightarrow \quad 4\pi \left[r^2 \frac{\partial G}{\partial r} \right]_0^1 = 1 \quad \Rightarrow \quad k = \frac{1}{4\pi}$$

So $$\boxed{G(r) = \frac{-1}{4\pi r}} \quad \Rightarrow \quad G(r, r') = \frac{-1}{4\pi |r-r'|}$$

Example of application

The gravitational potential of a galaxy made of stars of density $\rho(\underline{r})$ is the solution of

$$\nabla^2 \Phi = 4\pi G \rho(\underline{r})$$

$$\Rightarrow \quad \Phi(\underline{r}) = \int_{\text{universe}} G(r, r') \cdot 4\pi G \rho(\underline{r}') \, d^3\underline{r}'$$

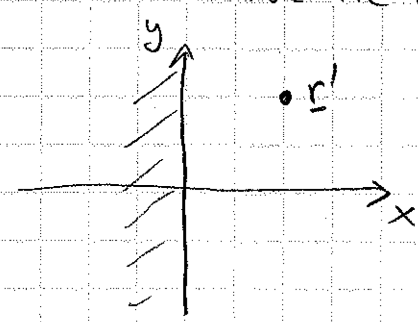
$$= \int_{\text{universe}} - \frac{\cancel{4\pi} G \rho(r') d^3 \underline{r}'}{\cancel{4\pi} |r - r'|} = \underbrace{-\int \frac{G\rho(r') d^3 r'}{|r-r'|}}_{\text{should be familiar....}}$$

③ Greens functions in non-infinite domains

(a) The method of images for semi-infinite domains

This method, beyond the interesting "trick" used, is a good illustration of a more general principle we shall derive later.

Here we consider a domain D, say $\{x \geq 0\}$, with $y \in \mathbb{R}$.



→ What is the Green's function for the Laplacian, with homogeneous Dirichlet condition on $x=0$, with a δ at $\underline{r}'$?

We could try to solve directly for $\nabla^2 G = \delta(\underline{r} - \underline{r}')$ and $G = 0$ on $x = 0$ – However, this is actually quite tricky ... It's actually much easier to construct G by using the Green's funct° in the ∞ domain (see previous section.

Let Γ be the solution of $\nabla^2 \Gamma = \delta(r - \underline{r}')$ in the ∞ domain. We know

$$\Gamma(\underline{r}, \underline{r}') = \frac{1}{2\pi} \ln(|\underline{r} - \underline{r}'|)$$

By construction $\nabla^2 \Gamma$ is zero everywhere except at the singularity $\underline{r}'$

Now Γ cannot be G, because $\Gamma \neq 0$ on the $x=0$ axis.

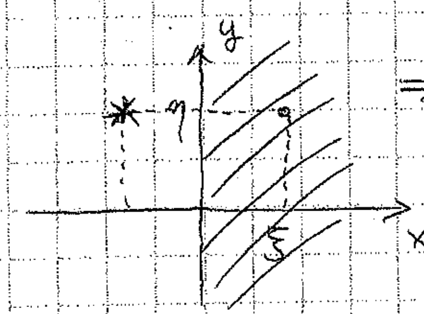
However, if we construct G as

$G = \Gamma + h$, then h must satisfy

$$\begin{cases} \nabla^2 h = 0 & \text{everywhere for } x > 0 \\ h = -\Gamma & \text{on } x = 0. \end{cases}$$

Method of images (reflection principle)

Idea: Construct the function h to be the symmetric function of Γ across the domain boundary:



$\Rightarrow$ If Γ has a pole in (x', y'), construct a function that has a pole in $(-x', y')$

here $h = -\Gamma(x, y; -x', y')$

so that $G = \Gamma(x, y; x', y') - \Gamma(x, y; -x' y')$

We can verify that

$\nabla^2 G = \nabla^2 \Gamma = \delta(r - \underline{r}')$ (since $\nabla^2 h = 0$ everywhere in D (the pole is outside of D))

$G(x=0) = 0$

since $h(0, y, \xi, \eta) = -\Gamma(0, y; \xi, \eta)$

$\left(\Gamma(x, y, x'; y') = \frac{1}{2\pi} \ln \sqrt{(x-x')^2 + (y-y')^2} \right.$ and $\Gamma(x, y, -x', y') = \frac{1}{2\pi} \ln \sqrt{(x+x')^2 + (y-y')^2}$

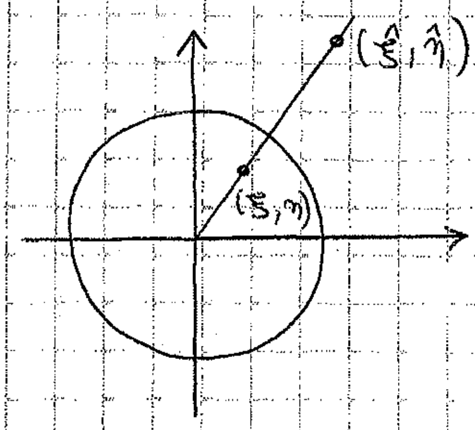
are equal at $x=0$)

(b) A "similar" construction can be used to compute h in a disk.

Consider the Dirichlet problem in the disk

$$\begin{cases} \nabla^2 \psi = f(x,y) & (x^2+y^2)^{1/2} < R \\ \psi(x,y) = g(x,y) & (x^2+y^2)^{1/2} = R \end{cases}$$

$\Rightarrow$ we want to find, for each (ξ,η), the function $h(x,y;\xi,\eta)$ such that



$$\begin{cases} \nabla^2 h = 0 & \text{in the disk} \\ h(x,y;\xi,\eta) = \Gamma(x,y;\xi,\eta) & \text{on the disk} \end{cases}$$

Trick: Consider the "inverse" point of (ξ,η), $(\hat{\xi},\hat{\eta})$ defined as

$$\begin{pmatrix} \hat{\xi} \\ \hat{\eta} \end{pmatrix} = \frac{R^2}{\xi^2+\eta^2} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

and let

$$h(x,y,\xi,\eta) = \Gamma\left[\frac{\sqrt{\xi^2+\eta^2}}{R} x, \frac{\sqrt{\xi^2+\eta^2}}{R} y; \frac{\sqrt{\xi^2+\eta^2}}{R} \hat{\xi}, \frac{\sqrt{\xi^2+\eta^2}}{R} \hat{\eta} \right]$$

$$= \Gamma\left[\frac{\sqrt{\xi^2+\eta^2}}{R} x, \frac{\sqrt{\xi^2+\eta^2}}{R} y; \frac{R}{\sqrt{\xi^2+\eta^2}} \xi, \frac{R}{\sqrt{\xi^2+\eta^2}} \eta \right]$$

Check:

(1) $\nabla^2 h = h_{xx} + h_{yy} = \frac{\xi^2+\eta^2}{R^2} \nabla^2 \Gamma = 0$

(2) $h(x,y;\xi,\eta) = -\frac{1}{2\pi} \ln\left[\frac{\xi^2+\eta^2}{R^2}(x-\hat{\xi})^2 + \frac{\xi^2+\eta^2}{R^2}(y-\hat{\eta})^2 \right]$

should be equal to

$\Gamma(x,y;\xi,\eta) = -\frac{1}{2\pi} \ln\left[(x-\xi)^2 + (y-\eta)^2\right]$

on the circle $x^2+y^2=R^2$

$$
\begin{aligned}
&\frac{\xi^2+\eta^2}{R^2}\left[(x-\hat{\xi})^2 + (y-\hat{\eta})^2\right] \\
&= \frac{\xi^2+\eta^2}{R^2}\left[x^2+y^2 - 2x\hat{\xi} - 2y\hat{\eta} + \hat{\xi}^2+\hat{\eta}^2\right] \\
&= (\xi^2+\eta^2) - (2x\xi + 2y\eta) + \frac{R^2}{\xi^2+\eta^2}(\xi^2+\eta^2) \\
&= R^2 - 2x\xi + 2y\eta + (\xi^2+\eta^2) \\
&= (x-\xi)^2 + (y-\eta)^2 \qquad \square .
\end{aligned}
$$

$\Rightarrow$ The Green's function on the disk of radius R is

$$
\begin{aligned}
G(x,y;\xi,\eta) &= \Gamma(x,y;\xi,\eta) - \Gamma\left(\frac{\sqrt{\xi^2+\eta^2}}{R}x, \frac{\sqrt{\xi^2+\eta^2}}{R}y ; \frac{R}{\sqrt{\xi^2+\eta^2}}\xi, \frac{R}{\sqrt{\xi^2+\eta}}\right. \\
&= -\frac{1}{2\pi}\ln\left((x-\xi)^2+(y-\eta)^2\right) \\
&\quad + \frac{1}{2\pi}\ln\left[\left(\frac{\sqrt{\xi^2+\eta^2}}{R}x - \frac{R}{\sqrt{\xi^2+\eta^2}}\xi\right)^2 + \left(\frac{\sqrt{\xi^2+\eta^2}}{R}y - \frac{R}{\sqrt{\xi^2+\eta^2}}\eta\right)^2\right]
\end{aligned}
$$

and the solution to the Dirichlet problem

$$
\begin{cases} \nabla^2 u = f & \text{in } D \\ u = g & \text{on } D \end{cases}
\quad \text{is} \quad u(\xi,\eta) = -\iint_D G f \, dx dy - \int_{\partial D} g \frac{\partial G}{\partial r} \, \underline{dl}
$$

(r = radial coordin

④ General considerations

The idea of constructing a Green's function in a bounded domain using the infinite domain one is quite general, e.g.

Given a domain D, we have

$$\begin{cases} \nabla^2 G = \delta(\underline{r}-\underline{r}') \\ G = 0 \text{ on } \partial D \end{cases}$$

$$\iff \begin{cases} G = \Gamma + h \quad \text{where} \\ \nabla^2 \Gamma = \delta(r-\underline{r}') \qquad \nabla^2 h = 0 \\ h = -\Gamma \text{ on } \partial D. \end{cases}$$

This provides a way of transforming the problem of finding the solution to a δ-function forcing with homogeneous conditions, to a Laplace problem with non-homogeneous conditions

This is not often easy however. See "tricks" learned earlier.

For generalizations to non-homogeneous BCs, and Von-Neumann bcs, see Pinchover & Rubinstein textbook

⑤ Solutions by coordinate change.

Solutions in domains with more complicated geometries can often be found by finding a mapping which transforms the domain into a basic one (disk. lines, etc). Conformal mappings are very powerful for this.