(II): General Non-homogeneous equations; introduction to Green's functions

(skip) 1 Non homogeneous (regular) SL problems (ODEs)

Given the ODE $\frac{1}{r(x)}\left[(p(x)u')' + q(x)u\right] = F(x)$

with bcs $\begin{cases} \alpha u(a) + \beta u'(a) = 0 \\ \gamma u(b) + \delta u'(b) = 0 \end{cases}$

① Seek solutions of the homogeneous eigenvalue eq.

$$\frac{1}{r(x)}\left[(p(x)u')' + q(x)u\right] = -\lambda u$$

→ this yields the eigenfunctions $\{v_n\}$ and eigenvalues $\{\lambda_n\}$

② Write $F(x) = \sum_n b_n v_n(x)$

(with $b_n = \int_a^b r(x')F(x')v_n(x')\,dx'$, if the v_ns are properly normalized)

Then since we know that the solution can also be written as

$$u(x) = \sum_n a_n v_n(x)$$

we can write

$$\frac{1}{r(x)}\left[(p(x)u')' + q(x)u\right] = \sum_n -\lambda_n a_n v_n(x) = \sum_n b_n v_n(x)$$

and by identification, $a_n = -\frac{b_n}{\lambda_n}$

$$\Rightarrow u(x) = \sum_n -\frac{b_n}{\lambda_n} v_n(x) = -\sum_n \int_a^b \frac{r(x')}{\lambda_n} F(x') v_n(x') v_n(x)\,dx'$$

$$= \int_a^b G(x, x')F(x')\,dx'$$

where $G(x;x') = -\sum_n \frac{1}{\lambda_n} v_n(x') v_n(x) \, r(x')$

- $G(x;x')$ is called the Green's function of the S.L problem
- It only depends on the characteristics of the homogeneous problem ($\{v_n\}, \{\lambda_n\}$) but, when integrated through with the forcing term $F(x)$, yields the solution of the forced problem
- Note that if the $\{v_n\}$ are not normalized then

$$G(x;x') = -\sum_n \frac{1}{\|v_n\|^2} \frac{r(x')}{\lambda_n} v_n(x') v_n(x) .$$

$$\text{where } \|v_n\|^2 = \int_a^b r(x) v_n(x)^2 \, dx .$$

<u>Example</u> Consider

$$y'' + y = 3\sin(2\pi x) \qquad \begin{array}{l} y(0) = 0 \\ y(1) = 0 \end{array}$$

We seek the eigenfunctions of $y'' + y = -\lambda y$

$$\rightarrow \quad y'' + (1+\lambda) y = 0$$

$$\text{so} \quad y = \alpha \cos(\sqrt{1+\lambda}\, x) + \beta \sin(\sqrt{1+\lambda}\, x)$$

$$\text{with} \begin{cases} \alpha = 0 \\ \sqrt{1+\lambda_n} = n\pi \end{cases}$$

$$\Rightarrow \begin{cases} \lambda_n = n^2\pi^2 - 1 \\ v_n(x) = \sin(n\pi x) \end{cases}$$

$\Rightarrow$ The Green's function $G(x,x') = -\sum_n \frac{\sin(n\pi x)\sin(n\pi x')}{\lambda_n \|\sin n\pi x\|^2}$

$$= -\sum_n \frac{2}{n^2\pi^2 - 1} \sin(n\pi x) \sin(n\pi x')$$

so the solution to the problem is $y(x) = \int_0^1 G(x,x')F(x')dx'$

$$y(x) = \int_a^1 \sum_n \frac{3\sin(2\pi x')}{1-n^2\pi^2} 2\sin(n\pi x)\sin(n\pi x')\,dx'$$

$$= \frac{3}{1-4\pi^2}\sin(2\pi x)$$

2. Application to parabolic/hyperbolic PDES.

Now consider either $u_t - \frac{1}{r(x)}\left[(p(x)u')' + q(x)u\right] = F(x,t)$

or $u_{tt} - \frac{1}{r(x)}\left[(p(x)u')' + q(x)u\right] = F(x,t)$.

Idea: Solve the associated Sturm-Liouville problem of the homogeneous PDE

$$\frac{1}{r(x)}\left[(p(x)u')' + q(x)u\right] + \lambda u = 0$$

to find the eigenvalues and eigenfunctions

$$\{v_n\},\ \{\lambda_n\}$$

then expand

$$F(x,t) = \sum_n b_n(t)v_n(x)$$

(in this case, $b_n(t) = \int_a^b F(x,t)r(x)v_n(x)\,dx$)

Assume a solution of the form

$$u(x,t) = \sum_n a_n(t)v_n(x)$$

and try the ansatz into the equation:

Parabolic case: $\sum_n \dot{a}_n(t)v_n(x) - \frac{1}{r(x)}\left[\left(p(x)\sum_n a_n(t)v_n'(x)\right)' + q(x)\sum a_n(t)v_n(x)\right]$

$$= \sum b_n(t)v_n(x)$$

so that

$$\sum_n{}' \dot{a}_n(t)\, v_n(x) + \sum_n{}' \lambda_n a_n(t) v_n(x) = \sum_n b_n(t) v_n(x)$$

and (by orthogonality):

$$\dot{a}_n + \lambda_n a_n = b_n(t)$$

$\hookrightarrow$ integrating factor method:

$$\frac{d}{dt}\left(a_n e^{\lambda_n t}\right) = b_n(t)\, e^{\lambda_n t}$$

so $$a_n(t) e^{\lambda_n t} - a_n(0) = \int_0^t b_n(t') e^{\lambda_n t'}\, dt'$$

$$\Rightarrow a_n(t) = a_n(0) e^{-\lambda_n t} + e^{-\lambda_n t} \int_0^t b_n(t') e^{\lambda_n t'}\, dt'.$$

Putting it all together we find that

$$u(x,t) = \sum_n{}' a_n(0) e^{-\lambda_n t} v_n(x) + \int_0^t \sum_n{}' e^{-\lambda_n (t-t')} v_n(x)\, b_n(t')\, dt'$$

$$= \sum_n{}' a_n(0) e^{-\lambda_n t} v_n(x) + \int_0^t \int_a^b \sum_n{}' e^{-\lambda_n(t-t')} v_n(x) v_n(x') r(x') \; F(x',t')\, dx' dt'$$

So we can write

$$u(x,t) = \sum_n a_n(0) e^{-\lambda_n t} v_n(x) + \int_0^t \int_a^b G(x,t;x',t') F(x',t')\, dx' dt'$$

$$\text{with } G(x,t;x',t') = \sum_n{}' e^{-\lambda_n(t-t')} v_n(x) v_n(x') r(x')$$

Here G is called the Heat Equation Kernel.
$\hookrightarrow$ another example of a Green's function.

$\hookrightarrow$ u is the sum of
- the solution to the problem with no forcing
- \+ the weighted integral of $F(x,t)$ with the Green's function.

Example of the drunks exiting the pub.

Recall: $\frac{\partial p}{\partial t} = k\frac{\partial^2 p}{\partial x^2} + S(x,t)$

$$\begin{cases} p(x,0)=0 \\ \frac{\partial p}{\partial x} = 0 \quad \text{at } x = 0, L \\ S(x,t) = S_0 e^{-\frac{t}{c}} \delta(x-\frac{L}{2}) \quad \text{for } t>0 \end{cases}$$

(take $t_0 = 0$).

Homogeneous problem; separation of variables to get spatial eigenmodes $\Rightarrow$

$$\begin{cases} V_n(x) = \cos\left(\frac{n\pi x}{L}\right) \\ \lambda_n = \frac{n^2\pi^2}{L^2} \end{cases}$$

So, by the previous calculation, we have

$$u(x,t) = \sum_{n=0}^{\infty}{}' a_n(0)\, e^{-\lambda_n t} V_n(x) + \int_0^t \int_0^L S(x',t')\, G(x,x';t,t')\, dx'dt'$$

where $a_n(0)$ are obtained by fitting u to initial conditions

$$u(x,0) = \sum_{n=0}^{\infty}{}' a_n(0) V_n(x) = 0 \Rightarrow a_n(0) = 0$$

and where $G(x,x';t,t') = \sum_{n=0}^{\infty}{}' e^{-\lambda_n(t-t')} \frac{V_n(x')V_n(x)\, r(x)}{\|V_n^2\|}$ but $r(x)=1$

$$\Rightarrow u(x,t) = \int_0^t\int_0^L \sum_{n=0}^{\infty} S_0 e^{-\frac{t'}{c}} \delta\left(x'-\frac{L}{2}\right) e^{-\frac{n^2\pi^2}{L^2}(t-t')} \frac{1}{\|V_n^2\|} \cos\left(\frac{n\pi x'}{L}\right)\cos\left(\frac{n\pi x}{L}\right) dx'dt'$$

$$= \int_0^t \sum_{n=0}^{\infty} S_0 e^{-\frac{t'}{c}} e^{-\frac{n^2\pi^2}{L^2}(t-t')} \cos\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi x}{L}\right) dt' \cdot \frac{1}{\|V_n^2\|}$$

$$= \sum_{n=0}^{\infty} S_0 \cos\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi x}{L}\right) \frac{1}{\frac{n^2\pi^2}{L^2} - \frac{1}{c}} \left[e^{-\frac{t}{c}} - e^{-\frac{n^2\pi^2}{L^2}t} \right] \frac{1}{\|V_n^2\|}$$

Hyperbolic case : similarly

$$\sum_n \ddot{a}_n(t) v_n(x) + \sum \lambda_n a_n(t) v_n(x) = \sum b_n(t) v_n(x)$$

so by orthogonality

$$\ddot{a}_n(t) + \lambda_n a_n(t) = b_n(t)$$

This time we use the Laplace transform method:

$$s^2 \hat{a}_n - s a_n(0) - a_n'(0) + \lambda_n \hat{a}_n = \hat{b}_n(s)$$

$$\Rightarrow \quad \hat{a}_n(s) = \frac{\hat{b}_n(s) + s a_n(0) + a_n'(0)}{s^2 + \lambda_n}$$

Now, the λ_n are positive $\Rightarrow$ the inverse Laplace transform (see Maple) is

$$a_n(t) = \frac{1}{\sqrt{\lambda_n}} \int_0^t b_n(t') \sin\left(\sqrt{\lambda_n}(t-t')\right) dt'$$

$$+ a_n(0) \cos\left(\sqrt{\lambda_n}\, t\right) + a_n'(0) \frac{1}{\sqrt{\lambda_n}} \sin\left(\sqrt{\lambda_n}\, t\right)$$

$\Rightarrow$ The general solution of the problem becomes

$$u(x,t) = \sum_{n=0}^{\infty} v_n(x) \cdot \left[\frac{1}{\sqrt{\lambda_n}} \int_0^t b_n(t') \sin\left(\sqrt{\lambda_n}(t-t')\right) dt' \right.$$

$$\left. + a_n(0) \cos\left(\sqrt{\lambda_n}\, t\right) + a_n'(0) \frac{1}{\sqrt{\lambda_n}} \sin\left(\sqrt{\lambda_n}\, t\right) \right]$$

but with $b_n(t') = \int_a^b \frac{F(x',t') v_n(x') r(x')\, dx'}{\|v_n^2\|}$

we get

$$u(x,t) = \sum \left[a_n(0) \cos\left(\sqrt{\lambda_n}\, t\right) + a_n'(0) \frac{1}{\sqrt{\lambda_n}} \sin\left(\sqrt{\lambda_n}\, t\right) \right] \cdot v_n(x)$$

$$+ \int_0^t \int_a^b F(x',t') G(x,x';t,t')\, dx'\, dt'$$

with

$$G(x,x';t,t') = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\lambda_n}} \sin\left(\sqrt{\lambda_n}(t-t')\right) \frac{V_n(x)V_n(x')r(x')}{\|V_n^2\|}$$

↳ the wave kernel

Example of the bridge

Recall :

$$u_{tt} - c^2 u_{xx} = \sin\left(\frac{2\pi x}{L}\right)\cos(\omega t)$$

$u = 0$ at both ends

$u_t(x,0) = u(x,0) = 0$

$\Rightarrow$ Eigenmodes/values of spatial homogeneous pb:

$$\begin{cases} V_n(x) = \sin\left(\frac{n\pi x}{L}\right) \\ \lambda_n = \frac{n^2\pi^2c^2}{L^2} \end{cases}$$

then $$u(x,t) = \sum \left[a_n(0)\cos\left(\frac{n\pi ct}{L}\right) + a_n'(0)\frac{L}{n\pi c}\sin\left(\frac{n\pi ct}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)$$

$$+ \int_0^t\int_0^L F(x',t')\,G(x,x';t,t')\,dx'\,dt'$$

Fitting this to ICs $\Rightarrow$ $a_n(0) = a_n'(0) = 0$

$$u(x,t) = \int_0^t dt' \int_0^L dx' \sin\left(\frac{2\pi x'}{L}\right)\cos(\omega t') \sum_{n=0}^{\infty} \frac{L}{n\pi c}\sin\left(\frac{n\pi c}{L}(t-t')\right)\frac{\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x'}{L}\right)}{\frac{L}{2}}$$

$$= \int_0^t dt' \cos(\omega t')\frac{L}{2\pi c}\sin\left(\frac{2\pi c}{L}(t-t')\right)\sin\left(\frac{2\pi x}{L}\right)$$

$$= \int_0^t \frac{dt'}{2}\left[\sin\left(\omega t' + \frac{2\pi c}{L}(t-t')\right) - \sin\left(\omega t' - \frac{2\pi c}{L}(t-t')\right)\right]\sin\frac{2\pi x}{L}\frac{L}{2\pi c}$$

$$= \frac{1/2}{\omega + \frac{2\pi c}{L}}\left[\cos\left(\frac{2\pi ct}{L}\right) - \cos\omega t\right] - \frac{1/2}{\omega - \frac{2\pi c}{L}}\left[\cos\left(\frac{2\pi c}{L}\right) - \cos\omega t\right]\frac{L}{2\pi c} \cdot \sin\left(\frac{2\pi x}{L}\right)$$

$$= \frac{1}{\omega^2 - 4\pi^2c^2/L^2}\left[\cos\left(\frac{2\pi ct}{L}\right) - \cos\omega t\right]\sin\left(\frac{2\pi x}{L}\right) \checkmark$$

CHAPTER 8: Green's functions

In this chapter we generalize the notion of a Green's function. We will focus here on time-independent problems, for simplicity. We will show that the solution to non-homogeneous problems can often be written as the convolution of the "forcing" with a <u>Green's function</u>, the latter being unique to each PDE & boundary conditions.

We start with the 1D problem to illustrate properties of Greens functions & learn how to construct them.

(I) Green's functions in 1D

Let's consider examples of the kind

$$\mathcal{L}(u) = f(x) \quad \text{with} \quad \mathcal{L}(u) = (p(x)u_x)_x + q(x)u$$

and homogeneous boundary conditions

$$\begin{cases} \alpha u(a) + \beta u'(a) = 0 \\ \delta u(b) + \gamma u'(b) = 0 \end{cases}$$

(see 9.3.5 for non homogeneous bcs)

We will discuss 2 different ways of finding the function $G(x, x')$ such that the solution to the problem, $u(x)$, is

$$u(x) = \int_a^b G(x, x') f(x') \, dx'$$

(a) Eigenvalue/eigenfunction expansion

As before, let's consider the SL problem

$$(p(x)u_x)_x + q(x)u = -\lambda u \qquad \text{with the same BCs}$$

$\rightarrow$ this has an ∞ sequence of eigenvalues λ_n with corresponding eigenvectors $v_n(x)$

We know that we can write any function $u(x)$ satisfying the BCs as:

$$u(x) = \sum_n \alpha_n v_n(x)$$

$\Rightarrow$ Plugging into the equation, we get

$$\mathcal{L}\left(\sum_n \alpha_n v_n(x)\right) = \sum_n \alpha_n \mathcal{L}(v_n(x)) = \sum_n -\lambda_n \alpha_n v_n(x) = f(x)$$

By orthogonality,

$$\alpha_n = -\frac{\langle f(x), v_n(x)\rangle}{\lambda_n \langle v_n, v_n\rangle} = -\frac{\int_a^b f(x')v_n(x')dx'}{\lambda_n \int_a^b v_n^2(x')dx'}$$

(provided $\lambda_n \neq 0$)

So finally,

$$u(x) = \sum_n -v_n(x) \frac{\int_a^b f(x')v_n(x')dx'}{\lambda_n \int_a^b v_n^2(x')dx'}$$

$$= \int_a^b f(x') G(x,x')\, dx'$$

provided $G(x,x') = \sum_n -\dfrac{v_n(x)v_n(x')}{\lambda_n \|v_n\|^2}$

$\Rightarrow$ This is one possible method for constructing the Green's function.

__Example:__ What is the general solution to

$$\frac{d^2u}{dx^2} + \omega^2 u = f(x) \qquad u(0)=0 \quad u(L)=0 \; ?$$

- Eigenfunctions: $\dfrac{d^2u}{dx^2} + \omega^2 u = -\lambda u$

$$\Rightarrow v_n(x) = \left\{ \begin{matrix} \cos\left(\sqrt{\lambda+\omega^2}\, x\right) \\ \sin\left(\sqrt{\lambda+\omega^2}\, x\right) \end{matrix} \right\}$$

By bcs, we have to have $\sqrt{\lambda+\omega^2} = \dfrac{n\pi}{L}$

so $\lambda_n = \dfrac{n^2\pi^2}{L^2} - \omega^2$.

so $v_n(x) = \sin\left(\dfrac{n\pi x}{L}\right)$

$\Rightarrow$ as long as we are off resonnance ($\lambda_n \neq 0$), we can write

$$G(x,x') = \sum_n - \frac{\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x'}{L}\right)}{\left(\frac{n^2\pi^2}{L^2}-\omega^2\right)\frac{L}{2}}$$

$$= -\frac{2}{L}\sum_n \frac{\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x'}{L}\right)}{\frac{n^2\pi^2}{L^2}-\omega^2}.$$

The solutions to any possible forcing can be found by convolving G with the forcing... But what does G look like? (see maple file).

(b) <u>Solution of the δ-function forcing.</u>

Another interesting, and much more fundamental method comes from noticing that G is the solution of

$$(*) \quad \mathcal{L}(G(x,x')) = \delta(x-x') \quad \text{with the same bcs.}$$

Indeed, remember that $\mathcal{L}$ is a linear operator, so

$$\mathcal{L}\left(\int_a^b G(x,x')f(x')dx'\right)$$

$$= \int_a^b \mathcal{L}(G(x,x'))f(x')dx' \quad \leftarrow \text{since } \mathcal{L} \text{ operates on the variable } x, \text{ not } x'$$

$$= \int_a^b \delta(x-x')f(x')dx' \quad \leftarrow \text{if indeed } G \text{ is solution of } (*)$$

$$= f(x)$$

(Note that the δ-function is an even function so $\delta(x-x') = \delta(x'-x)$)

$\Rightarrow$ The problem is now shifted to: how to <u>find</u> solutions of

$$\mathcal{L}(G) = \delta(x-x').$$

In order to do this, note that $\delta(x-x')$ is effectively 0 unless $x = x'$ so we mostly solve

$$\mathcal{L}(G) = 0 \quad \text{except at } x = x'.$$

Since $\mathcal{L}$ is a 2nd order linear operator, the general solution which can be written as a linear combination of $\underline{\underline{2}}$ basic functions.

Hence $\mathcal{L}(G) = 0$

$$\Rightarrow \quad G(x,x') = \alpha u_1(x) + \beta u_2(x)$$

There will be one such solution on either side of the point x' $\Rightarrow$

On the left $(x<x')$ $\quad G_L(x;x') = \alpha_L u_1(x) + \beta_L u_2(x)$

On the right $(x>x')$ $\quad G_R(x;x') = \alpha_R u_1(x) + \beta_R u_2(x)$

Two of these 4 unknown coefficients can be found by fitting the boundary conditions, at $x=a$ and $x=b$.

The other 2 are found by requiring that

- $G(x,x')$ be <u>continuous</u> at $x=x'$
- The derivatives of $G(x,x')$ on either side of $x=x'$ satisfy the correct "jump" condition, obtainde by integrating (*) across $x=x'$:

$$\int_{x'-\epsilon}^{x'+\epsilon} \mathcal{L}(G)\,dx = \int_{x'-\epsilon}^{x'+\epsilon} \delta(x-x')\,dx = 1$$

$$\hookrightarrow \int_{x'-\epsilon}^{x'+\epsilon} \left[\left(p(x)G_x\right)_x + q(x)G\right] dx = 1$$

$$\hookrightarrow \lim_{\epsilon\to 0} \left\{ \left[p(x'+\epsilon)\, G_x\big|_{x'+\epsilon} - p(x'-\epsilon)\, G_x\big|_{x'-\epsilon} \right\} = 1$$

(since q, u continuous)

$\hookrightarrow \lim_{\epsilon \to 0} \left[\frac{dG}{dx}\Big|_{x=x'+\epsilon} - \frac{dG}{dx}\Big|_{x=x'-\epsilon} \right] = \frac{1}{p(x')}$

$\hookrightarrow \quad \alpha_R u_1'(x') + \beta_R u_2'(x') - \alpha_L u_1'(x) - \beta_L u_2'(x') = \frac{1}{p(x')}$

(The fourth condition).

While this method is very general, it is more easily understood when applied to a particular example.

<u>Example</u>: $\frac{d^2u}{dx^2} + \omega^2 u = f(x)$. (same as before)

We now seek $G(x, x')$, the solution of

$$\frac{d^2G}{dx^2} + \omega^2 G = \delta(x - x')$$

- The equation $\frac{d^2G}{dx^2} + \omega^2 G = 0$ has a general solutions of the kind $\quad G = \alpha \cos\omega x + \beta \sin\omega x$.

$\Rightarrow$ let's consider the 2 solutions on either side of x':

On left: $G_L(x) = \alpha_L \cos\omega x + \beta_L \sin\omega x$

From $u(0) = 0$ we get $G(0, x') = 0$ so
$G_L(0) = 0 \Rightarrow \alpha_L = 0$ so $G_L(x) = \beta_L \sin\omega x$

On right: $G_R(x) = \alpha_R \cos\omega x + \beta_R \sin\omega x$

From $u(L) = 0$ we get $G(L, x') = 0$ so
$G_R(L) = 0 \Rightarrow$

$$\alpha_R \cos(\omega L) + \beta_R \sin(\omega L) = 0$$

$$\Rightarrow \frac{\alpha_R}{\beta_R} = -\tan(\omega L) \Rightarrow \alpha_R = -\beta_R \tan(\omega L)$$

- From continuity at $x = x'$ we get

$$\beta_L \sin(\omega x') = -\beta_R \tan(\omega L)\cos(\omega x') + \beta_R \sin(\omega x')$$

So $\beta_L = \beta_R \left(1 - \frac{\tan\omega L}{\tan\omega x'}\right)$

- Finally, the derivative jump condition implies

$$\left.\frac{dG_R}{dx}\right|_{x'} - \left.\frac{dG_L}{dx}\right|_{x'} = 1 \qquad (\text{here, } p(x) = 1)$$

$$\Rightarrow \quad \beta_R \omega \tan\omega L \sin\omega x' + \beta_R \omega \cos\omega x' - \beta_L \omega \cos\omega x' = 1$$

So

$$\beta_R \left[\omega \tan\omega L \sin\omega x' + \omega\cos\omega x' - \omega\cos\omega x'\left(1 - \frac{\tan\omega L}{\tan\omega x'}\right)\right] = 1$$

$$\beta_R = \frac{1}{\omega\tan(\omega L)} \frac{1}{\sin\omega x' + \frac{\cos\omega x'}{\tan\omega x'}}$$

$$= \frac{\sin(\omega x')}{\omega\tan(\omega L)}$$

$$\beta_L = \frac{1}{\omega\tan(\omega L)}\left(\sin\omega x' - \tan\omega L \cos\omega x'\right)$$

$$= \frac{1}{\omega}\left(\frac{\sin\omega x'}{\tan\omega L} - \cos\omega x'\right)$$

$$\alpha_R = -\frac{\sin(\omega x')}{\omega}.$$

$$\Rightarrow \quad G(x,x') = \begin{cases} \frac{1}{\omega}\left(\frac{\sin\omega x'}{\tan\omega L} - \cos\omega x'\right)\sin\omega x & \text{if } x < x' \\ -\frac{\sin\omega x'}{\omega}\cos\omega x + \frac{\sin\omega x' \sin\omega x}{\omega\tan\omega L} & x > x' \end{cases}$$

It can be shown (most easily by plotting them, see maple) that this is exactly the same function as before.

(c) Variation of constants (variation of parameters)

For completeness note that there is another method for solving forced linear ODES, which however (to my knowledge) does not carry over in higher dimensions.

see RHB 15.2.4 for detail (our textbook is not great for that).

Given $\mathcal{L}(u) = f(x)$

- The general solution of the homogeneous equation $\mathcal{L}(u)=0$ is

$$u_G(x) = \alpha_1 u_1(x) + \alpha_2 u_2(x)$$

- We seek a solution of the forced equation as

$$u_p(x) = \beta_1(x)\, u_1(x) + \beta_2(x)\, u_2(x)$$

i.e. we "vary" the constants

where $\beta_1(x)$ and $\beta_2(x)$ are TBD.

- The original equation, $\mathcal{L}(u_p) = f(x)$, provides one constraint on the 2 functions β_1 and β_2. We are free to select a second constraint; we do so in a way which simplifies the algebra required in solving for β_1 and β_2.

Trick: Require that $\boxed{\beta_1'(x)\, u_1(x) + \beta_2'(x)\, u_1(x) = 0}$.

(see RHB for generalizations to higher-order ODES).

Why does this help?

Note: ' notation and $_x$ notation used interchangably here

$$\mathcal{L}(u_p) = f(x) \quad \Rightarrow \quad \left(p(x)\, u_{px}\right)_x + q(x)\, u_p = f(x)$$

$$\Rightarrow \left[p(x)\left[\beta_1 u_1 + \beta_2 u_2\right]_x\right]_x + q(x)\left[\beta_1 u_1 + \beta_2 u_2\right] = f(x)$$

$$\Rightarrow \left[p(x)\left\{\underbrace{\beta_1' u_1 + \cancel{\beta_2' u_2}}_{0 \text{ by other constraint}} + \beta_1 u_{1x} + \beta_2 u_{2x}\right\}\right]_x + q\left(\beta_1 u_1 + \beta_2 u_2\right) = f(x)$$

$$\Rightarrow \quad \underbrace{\beta_1 (p u_{1x})_x + \beta_2 (p u_{2x})_x}_{} + \beta_{1x} p u_{1x} + \beta_{2x} p u_{2x}$$

$$\underbrace{+ q\beta_1 u_1 + q\beta_2 u_2}_{=0 \text{ since } u_1, u_2 \text{ are solutions of } \mathcal{L}(u)=0.} = f(x)$$

$$\Rightarrow \quad \boxed{\beta_1' u_1' + \beta_2' u_2' = \frac{f(x)}{p(x)}}$$

So now we have 2 equations for two functions β_1' and β_2', which can be integrated to get $\beta_1(x)$ and $\beta_2(x)$.

Example $\quad \dfrac{d^2u}{dx^2} + \omega^2 u = f(x).$ with $u(0)=0$ $u(L)=0$.

$\rightarrow$ The two general homogeneous solutions are $\cos\omega x$ and $\sin\omega x$ $\rightarrow$ we seek

$$u_p(x) = \beta_1(x)\cos\omega x + \beta_2(x)\sin\omega x.$$

$\rightarrow$ We know that

$$\begin{cases} \beta_1' \cos\omega x + \beta_2' \sin\omega x = 0 \\ -\beta_1' \omega \sin\omega x + \omega\beta_2' \cos\omega x = f(x) \end{cases}$$

$$\Rightarrow \begin{cases} \beta_1' = -\dfrac{f(x)\sin\omega x}{\omega} \\[2ex] \beta_2' = \dfrac{f(x)}{\omega}\cos\omega x \end{cases}$$

$$\Rightarrow \text{So} \quad \beta_1(x) = -\int_0^x \frac{f(x')\sin\omega x'}{\omega}\,dx' + \beta_1(0)$$

$$\beta_2(x) = +\int_0^x \frac{f(x')\cos\omega x'}{\omega}\,dx' + \beta_2(0)$$

To fit the BCs we want $u(0)=0$ and $u(L)=0$

$$u(0) = 0 = \beta_1(0)\,u_1(0) + \beta_2(0)u_2(0)$$
$$= \beta_1(0) \quad \Rightarrow \quad \beta_1(0) = 0$$

$$u(L) = 0 = \beta_1(L)\,u_1(L) + \beta_2(L)\,u_2(L)$$
$$= -\int_0^L \frac{f(x')\sin\omega x'}{\omega}\,dx' \cdot \cos\omega L$$
$$+ \left[\int_0^L \frac{f(x')\cos\omega x'}{\omega}\,dx' + \beta_2(0)\right]\sin\omega L$$
$$\rightarrow \quad \beta_2(0) = -\int_0^L \frac{f(x')\cos\omega x'}{\omega}\,dx'$$
$$+ \int_0^L \frac{f(x')\sin\omega x'}{\omega\tan\omega L}\,dx'$$

$$\Rightarrow \beta_1(x) = -\int_0^x f(x')\,\frac{\sin\omega x'}{\omega}\,dx'$$

$$\beta_2(x) = -\int_x^L \frac{f(x')\cos\omega x'}{\omega}\,dx' + \int_0^L \frac{f(x')\sin\omega x'\,dx'}{\omega\tan\omega L}.$$

$$\Rightarrow u(x) = -\cos\omega x \int_0^x f(x')\,\frac{\sin\omega x'}{\omega}\,dx'$$
$$+ \sin\omega x\left\{-\int_x^L \frac{f(x')\cos\omega x'}{\omega}\,dx' + \int_0^L \frac{f(x')\sin\omega x'\,dx'}{\omega\tan\omega L}\right\}$$

$$= \int_0^L G(x,x')\,dx' \quad \text{provided}$$

$$G(x,x') = \begin{cases} -\dfrac{\cos\omega x\sin\omega x'}{\omega} + \dfrac{\sin\omega x'\sin\omega x}{\omega\tan\omega L} & x' < x \\[2ex] -\dfrac{\sin\omega x\cos\omega x'}{\omega} + \dfrac{\sin\omega x'\sin\omega x}{\omega\tan\omega L} & x' > x \end{cases}$$

as before