Aside

a probabilistic derivation of the diffusion equation (Brownian motion)

Imagine a lattice (in 1D). Define the concentration $C(x,t)$ as the expected number of particles at position x, time t

Particles have equal probability to move left or right (p) and probability to stay where they are $(1-2p)$

So $$C(x,t+\Delta t) = p\left(C(x-\Delta x,t) + C(x+\Delta x,t)\right) + (1-2p)\,C(x,t)$$

Now assume Δt small and Δx small then

$$C(x,t) + \Delta t\,\frac{\partial c}{\partial t} = p\left[2c(x,t) + \Delta x^2\,\frac{\partial^2 c}{\partial x^2}\right] + (1-2p)c(x,t)$$
$$= c(x,t) + p\Delta x^2\,\frac{\partial^2 c}{\partial x^2}$$

$$\Rightarrow \quad \frac{\partial c}{\partial t} = p\,\frac{\Delta x^2}{\Delta t}\,\frac{\partial^2 c}{\partial x^2}$$

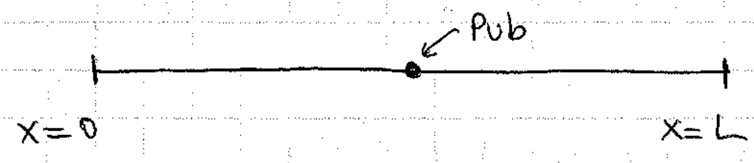
define this as the diffusion coefficient k.

Note 1: In the presence of forces this derivation leads to Fokker-Planck equation.

Note 2 From more general considerations, you can derive all the PDEs of fluid mechanics from statistical averaging of ensemble properties of individual particles. Kinetic theory (Boltzmann's equation and its moments).

② Forced diffusion equation

A pub in England rings last orders at 11:00 pm, at which point people start to leave and go back home. They are only "locals", i.e. people living in the same 1D street. Being quite drunk, they walk randomly in the street although they dont leave it. We assume they can't find their keys and stay in the street a long time ...



← Street of length L.

Pub at $x=\frac{L}{2}$

⊕ We model the evolution of the population density in the street as a diffusion process:

$$\Rightarrow \quad \frac{\partial p}{\partial t} = k\frac{\partial^2 p}{\partial x^2} + S(x,t)$$

- population density $p = p(x,t)$.
- $S(x,t) =$ # of people / unit time being released in the street by the pub.

⊕ At $t = t_0$ the street is empty

⊕ To model the "they don't leave the street" idea, we use <u>insulating</u> boundary conditions.

$$\Rightarrow \quad \frac{\partial p}{\partial x} = 0 \quad \text{at both boundaries.}$$

⊕ To model the flux of people out of the pub, we assume

$$S(x,t) = S_0 e^{-(t-t_0)/\tau} \, \delta\left(x - \frac{L}{2}\right)$$

$t_0 =$ last orders time

$\tau \simeq$ time till closing, say 1/2 hour.

$\delta =$ a delta function

Recall: $\int_{-\infty}^{\infty} \delta(x-a) f(x)\, dx = f(a)$.

Solution. 1. Find the spatial eigenmodes of the homogeneous problem.

- From previous lectures, we know that

$$\begin{cases} A_0(x) = a_0 x + b_0 \\ A_n(x) = a_n \cos(\lambda_n x) + b_n \sin(\lambda_n x) \end{cases} \quad (\lambda \text{ to be determined})$$

to satisfy $\frac{dp}{dx} = 0$ at both ends we need

- $n \neq 0$ $\quad \frac{dA_n}{dx} = \lambda_n\left(-a_n \sin(\lambda_n x) + b_n \cos(\lambda_n x)\right)$

$$\Rightarrow \begin{cases} \left.\frac{dA_n}{dx}\right|_{x=0} = 0 \Rightarrow b_n = 0 \\ \left.\frac{dA_n}{dx}\right|_{x=L} = 0 \Rightarrow \lambda_n = \frac{n\pi}{L} \end{cases} \Rightarrow A_n(x) = \cos\left(\frac{n\pi x}{L}\right)$$

(ignore constant)

- $n = 0$ $\quad A_0(x) = \text{constant} = 1$ (ignore constant).
  ↳ can be written as $\cos\left(\frac{0\pi x}{L}\right)$.

3. Note that $\int_0^L A_n(x) A_m(x)\, dx = \frac{L}{2}\delta_{mn} + \frac{L}{2}\delta_{m0}\delta_{mn}$

2. Suppose the solution is

$$p(x,t) = \sum_0^\infty A_n(x) B_n(t) \quad \text{and plug into PDE}$$

$$\Rightarrow \sum_0^\infty A_n(x)\dot{B}_n(t) = k\sum_0^\infty -\frac{n^2\pi^2}{L^2} A_n(x) B_n(t) + S(x,t)$$

multiply by $A_m(x)$, integrate on $[0,L]$ ...

- $m \neq 0$ $\quad \frac{L}{2}\dot{B}_m(t) = -\frac{m^2\pi^2 k}{L^2}\cdot\frac{L}{2} B_m(t) + \int_0^L S(x,t) A_m(x)\, dx$

Now $\int_0^L A_m(x) S_0 e^{-(t-t_0)/\tau} \delta(x - \frac{L}{2})\, dx$

$$= S_0 e^{-(t-t_0)/\tau} A_m\left(\frac{L}{2}\right) = S_0 e^{-(t-t_0)/\tau} \cos\left(\frac{m\pi}{2}\right)$$

- $m = 0$: $\quad L\dot{B}_0(t) = +S_0 e^{-\frac{t-t_0}{\tau}} \Rightarrow B_0(t) = h$

$\Rightarrow$ The set of decoupled ODES for the B_n are

$$\dot{B}_n + \frac{n^2\pi^2 k}{L^2} B_n = \frac{2}{L} S_0 e^{-(t-t_0)/\tau} \cos\left(\frac{n\pi}{2}\right)$$

$\rightarrow$ the general solution of the homogeneous problem is

$$B_n^G(t) = \alpha_n e^{-\frac{n^2\pi^2 k}{L^2} t}$$

The particular solution: try $B_n^{PS}(t) = K e^{-\frac{t-t_0}{\tau}}$

$$\Rightarrow \quad -\frac{1}{\tau} K + \frac{n^2\pi^2 k}{L^2} K = \frac{2}{L} S_0 \cos\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow \quad K = \frac{\frac{2}{L} S_0 \cos\left(\frac{n\pi}{2}\right)}{\frac{n^2\pi^2 k}{L^2} - \frac{1}{\tau}}$$

So finally, we have $B_n(t) = \alpha_n e^{-\frac{n^2\pi^2 k}{L^2} t} + \dfrac{\frac{2}{L} S_0 \cos\left(\frac{n\pi}{2}\right)}{\frac{n^2\pi^2 k}{L^2} - \frac{1}{\tau}} e^{-\frac{(t-t_0)}{\tau}}$
for $(n \neq 0)$

$\Rightarrow$ $p(x,t) = \sum_{n=0}^{\infty} A_n(x) B_n(t)$ is the complete solution, where the α_ns remain to be determined.

At $t = t_0$ $p(x,t) = 0$ (the street is empty before $t = t_0$)

$$\Rightarrow \quad \sum_{n=0}^{\infty} A_n(x) B_n(0) = 0$$

$$b_0 - \frac{\tau}{L} S_0 e^{-\frac{t-t_0}{\tau}} + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi x}{L}\right)\left[\alpha_n e^{-\frac{n^2\pi^2 k t_0}{L^2}} + \frac{\frac{2}{L} S_0 \cos\left(\frac{n\pi}{2}\right)}{\frac{n^2\pi^2 k}{L^2} - \frac{1}{\tau}}\right] = 0$$

$$\Rightarrow \quad \alpha_n = \frac{-S_0 \cos\left(\frac{n\pi}{2}\right) \cdot \frac{2}{L}}{\frac{n^2\pi^2 k}{L^2} - \frac{1}{\tau}} e^{\frac{n^2\pi^2 k t_0}{L^2}} \quad \text{and} \quad b_0 = \frac{\tau}{L} S_0$$

$$\Rightarrow p(x,t) = \sum_{n=1}^{\infty} \cos\left(\frac{n\pi x}{L}\right) \frac{\frac{2}{L} S_0 \cos\left(\frac{n\pi}{2}\right)}{\frac{n^2\pi^2 k}{L^2} - \frac{1}{\tau}} \left(-e^{-\frac{n^2\pi^2}{L} k (t-t_0)} + e^{-(t-t_0)/\tau} \right)$$

$$+ \frac{\tau}{L} S_0 \left(1 - e^{-\frac{t-t_0}{\tau}}\right)$$

Note. ① The total number of people in the street at any time is easily derived from the PDE

$$\Rightarrow \qquad \frac{\partial p}{\partial t} = k\frac{\partial^2 p}{\partial x^2} + S(x,t)$$

$$\hookrightarrow \frac{\partial}{\partial t}\int_0^L p(x,t)\,dx = k\int_0^L \frac{\partial^2 p}{\partial x^2}\,dx + \int_0^L S(x,t)\,dx$$

$$= k\left[\frac{\partial p}{\partial x}\right]_0^L + S_0 e^{-\frac{t-t_0}{\tau}}$$

$$= S_0 e^{-\frac{t-t_0}{\tau}}$$

So

$$\int_0^L p(x,t)\,dx = \int_{t_0}^{t} S_0 e^{-\frac{t'-t_0}{\tau}}\,dt'$$

$$= \tau S_0\left[1 - e^{-\frac{t-t_0}{\tau}}\right] \quad (t > t_0)$$

$\hookrightarrow$ at any time the # of people in the street is equal to the total # which has left the pub already as expected.

② See movies:

– if $\tau \ll \frac{L^2}{\pi^2 k}$ then a large # of people are rapidly released, and then diffuse away from pub entrance

– if $\tau \gg \frac{L^2}{\pi^2 k}$ then the diffusion is faster than release & the people are always ~ evenly spread in the street.

③ Note the "resonnance" between $\frac{1}{\tau}$ and $\frac{n^2\pi^2 k}{L^2}$

$\Rightarrow$ if $\tau \ll \frac{L^2}{\pi k}$ then $\exists$ n such that $\frac{1}{\tau} \simeq \frac{n^2\pi^2 k}{L^2}$

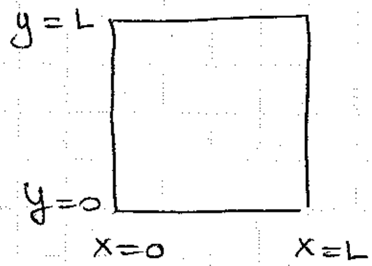
That n determines the typical initial "width" of the people density function. (See movie) as $\Delta = \frac{L}{n\pi}$

③ Poisson equation

Suppose we want to solve $\nabla^2 T = -H(x,y)$

↑ heating source

to obtain the steady-state temperature profile in a metallic plate, heated as prescribed by $H(x,y)$ and with $T=0$ on all 4 sides; take $k=1$



Note that the − sign comes from

$$\frac{\partial T}{\partial t} = \nabla^2 T + H(x,y)$$

→ in steady state $\nabla^2 T = -H$

The spatial eigenmodes in x-direction are (see previous lectures), for $T(0,y) = T(L,y) = 0$

$$A_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

→ Assume $\quad T(x,y) = \sum_{n=1}^{\infty} A_n(x)\, B_n(y)$

Then

$$\sum_{n=1}^{\infty} -\frac{n^2\pi^2}{L^2} A_n(x)\, B_n(y) + A_n(x)\frac{d^2B_n}{dy^2} = -H(x,y)$$

Noting that $\quad \int_0^L A_n(x) A_m(x)\, dx = \frac{L}{2}\delta_{mn}$,

$$\frac{d^2B_n}{dy^2} - \frac{n^2\pi^2}{L^2} B_n = -\frac{2}{L}\int_0^L H(x,y)\sin\left(\frac{n\pi x}{L}\right) dx.$$

$$= -h_n(y).$$

Suppose that $\quad H(x,y) = \delta\left(x-\frac{L}{2}\right)\delta\left(y-\frac{L}{2}\right)$
to model a point source

Then $$\frac{d^2 B_n}{dy^2} - \frac{n^2\pi^2}{L^2} B_n = -\sin\left(\frac{n\pi}{2}\right) \delta\left(y - \frac{L}{2}\right) \cdot \frac{2}{L}$$

Laplace transforms, review

Laplace transforms are very useful for solving non-homogeneous linear ODEs.

Idea: let f be a function of t

The Laplace transform of f is

$$\mathcal{L}(f) = \hat{f}(p) = \int_0^\infty f(t) e^{-pt}\, dt.$$

Properties: $\mathcal{L}(f') = p\hat{f}(p) - f(0)$

since $$\int_0^\infty \frac{df}{dt} e^{-pt}\, dt = \left[f e^{-pt} \right]_0^\infty + \int_0^\infty p f e^{-pt} dt$$
$$= p\hat{f}(p) - f(0)$$

$$\mathcal{L}(f'') = p^2 \hat{f}(p) - p f(0) - f'(0)$$
(proof is similar).

So given a linear ODE with constant coefficients

$$a f'' + b f' + c f = g(t) \qquad (*)$$

$$\mathcal{L}(*) \Rightarrow \quad a\left[p^2 \hat{f}(p) - p f(0) - f'(0) \right]$$
$$+ \quad b\left[p\hat{f}(p) - f(0) \right]$$
$$+ \quad c\hat{f}(p) = \int_0^\infty g(t) e^{-pt}\, dt = G(p)$$

Suppose $f(0)$ and $f'(0)$ are known (initial value problem) then this is an algebraic equation for $\hat{f}(p)$.

To recover $f(t)$, we need to do an inverse Laplace transform.

For detail on Inverse Laplace transforms, see handout. Usually, it's easy to find the solution using Inverse Laplace transform tables.

Here:

$$\frac{d^2 B_n}{dy^2} - \frac{n^2\pi^2}{L^2} B_n = -\sin\left(\frac{n\pi}{2}\right) \delta\left(y - \frac{L}{2}\right) \cdot \frac{2}{L}$$

$$\Rightarrow \quad p^2 \hat{B}_n - p B_n(0) - B_n'(0)$$

$$- \frac{n^2\pi^2}{L^2} \hat{B}_n = -\frac{2}{L} \sin\left(\frac{n\pi}{2}\right) \int_0^\infty \delta\left(y - \frac{L}{2}\right) e^{-py}\, dy$$

$$= -\sin\left(\frac{n\pi}{2}\right) e^{-p\frac{L}{2}} \cdot \frac{2}{L}$$

$B_n(0) = 0$ but $B_n'(0)$ is unknown. Let's leave it as is for the moment.

$$\Rightarrow \quad \hat{B}_n \left[p^2 - \frac{n^2\pi^2}{L^2} \right] = B_n'(0) - \frac{2}{L} \sin\left(\frac{n\pi}{2}\right) e^{-p\frac{L}{2}}$$

$$\text{so} \quad \hat{B}_n(p) = \frac{B_n'(0)}{p^2 - \frac{n^2\pi^2}{L^2}} - \frac{2}{L} \sin\left(\frac{n\pi}{2}\right) e^{-p\frac{L}{2}} \frac{1}{p^2 - \frac{n^2\pi^2}{L^2}}$$

From tables:

- The inverse transform of $\frac{1}{p^2 - a^2}$ is $\frac{\sinh(ay)}{a}$
- The inverse transform of

$$\frac{e^{-pb}}{p^2 - a^2} \quad \text{is} \quad \begin{cases} \frac{\sinh(a(y-b))}{a} & \text{if } y > b \\ 0 & \text{if } 0 < y < b \end{cases}$$

$$\Rightarrow \quad B_n(y) = \frac{B_n'(0)}{\frac{n\pi}{L}} \cdot \sinh\left[\frac{n\pi y}{L}\right] - \frac{2}{L}\sin\left(\frac{n\pi}{2}\right)\sinh\left(\frac{n\pi(y-\frac{L}{2})}{L}\right)\frac{1}{\frac{n\pi}{L}} \quad \text{if } y > \frac{L}{2}$$

At $y = L$, the solution is such that $B_n(L) = 0 \Rightarrow$

$$B_n'(0)\sinh(n\pi) - \frac{2}{L}\sin\left(\frac{n\pi}{2}\right)\sinh\left(\frac{n\pi}{2}\right) = 0$$

$$\Rightarrow \quad B_n'(0) = \frac{2}{L}\frac{\sin\left(\frac{n\pi}{2}\right)\sinh\left(\frac{n\pi}{2}\right)}{\sinh(n\pi)}.$$

So finally, we have

$$T(x,y) = \sum_{n=1}^{\infty}{}' \sin\left(\frac{n\pi x}{L}\right)\frac{\sin\left(\frac{n\pi}{2}\right)}{\frac{n\pi}{L}}\frac{2}{L}\left[\frac{\sinh\left(\frac{n\pi}{2}\right)}{\sinh(n\pi)}\sinh\left(\frac{n\pi y}{L}\right) - \sinh\left(\frac{n\pi(y-\frac{L}{2})}{L}\right) H\left(y-\frac{L}{2}\right)\right]$$

Heaviside function

Note: This expression is slightly awkward, but it can be shown that it indeed leads to the correct behavior in y, which should be symmetry across the $y = \frac{L}{2}$ line. (Homework!)