AMS 212A:
Partial Differential Equations

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Course information

- Everything you need to know about the course can be found on the course web-site:

  http://www.cse.ucsc.edu/classes/ams212a/Winter11/index.html

Course information

- This class will teach you to:
  - Understand the nature of PDEs, and the expected behavior of the solutions, simply by inspection.
  - Solve certain classes of PDEs analytically using a range of different techniques.
  - Improve your mathematical modeling skills, from the creation of a model to the critical analysis of the solutions.

- This class do *not* cover numerical solutions of PDEs (see AMS213)
- This class is an applied class and does *not* address finer details of existence of solutions (see MATH classes).

Examples of PDEs.

1. The wave equation

Waves/oscillations

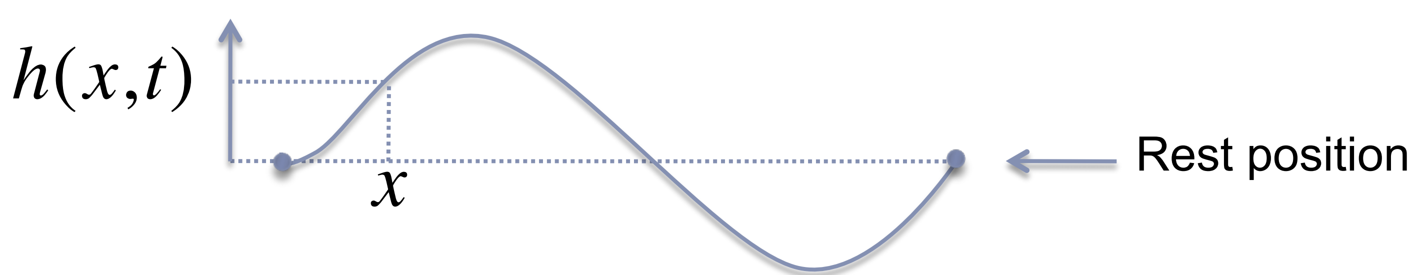
- The wave equation:

$$\frac{\partial^2 f}{\partial t^2} = c^2 \nabla^2 f$$

Wave speed square

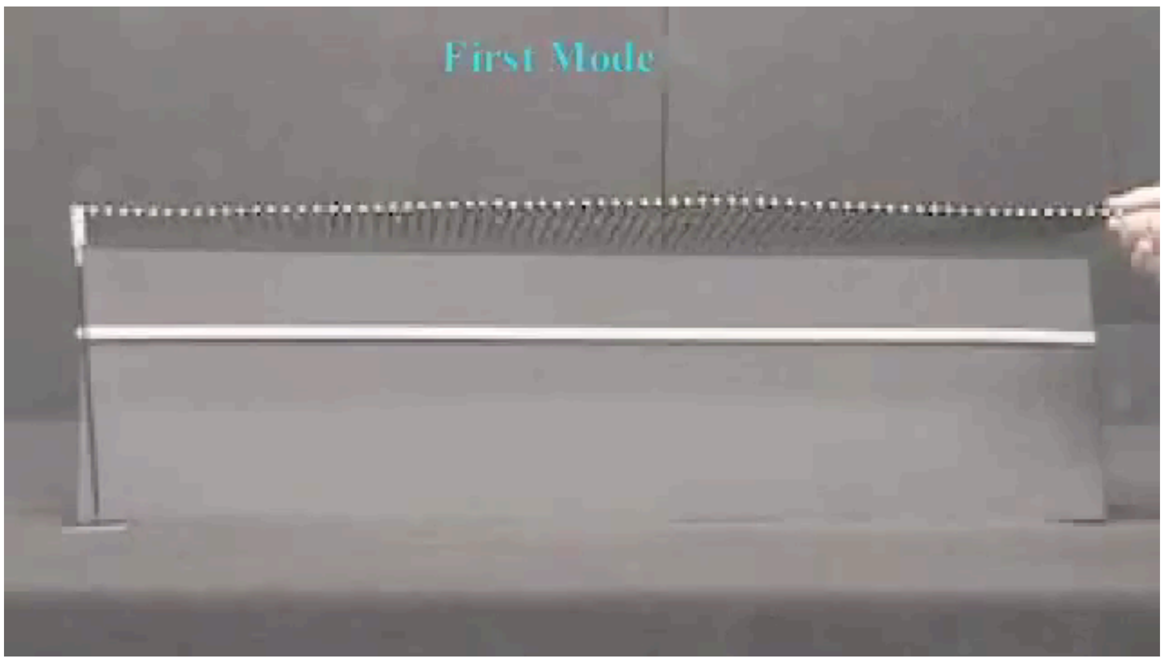
- Examples:
  - Displacement waves of a string under tension (e.g. guitar string)

$$\frac{\partial^2 h}{\partial t^2} = c^2 \frac{\partial^2 h}{\partial x^2}$$



Waves/oscillations

- Note that different behaviors are possible:
  - standing waves (oscillations) with different possible modes



Waves/oscillations

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  - propagating waves, with the possibility of reflections.

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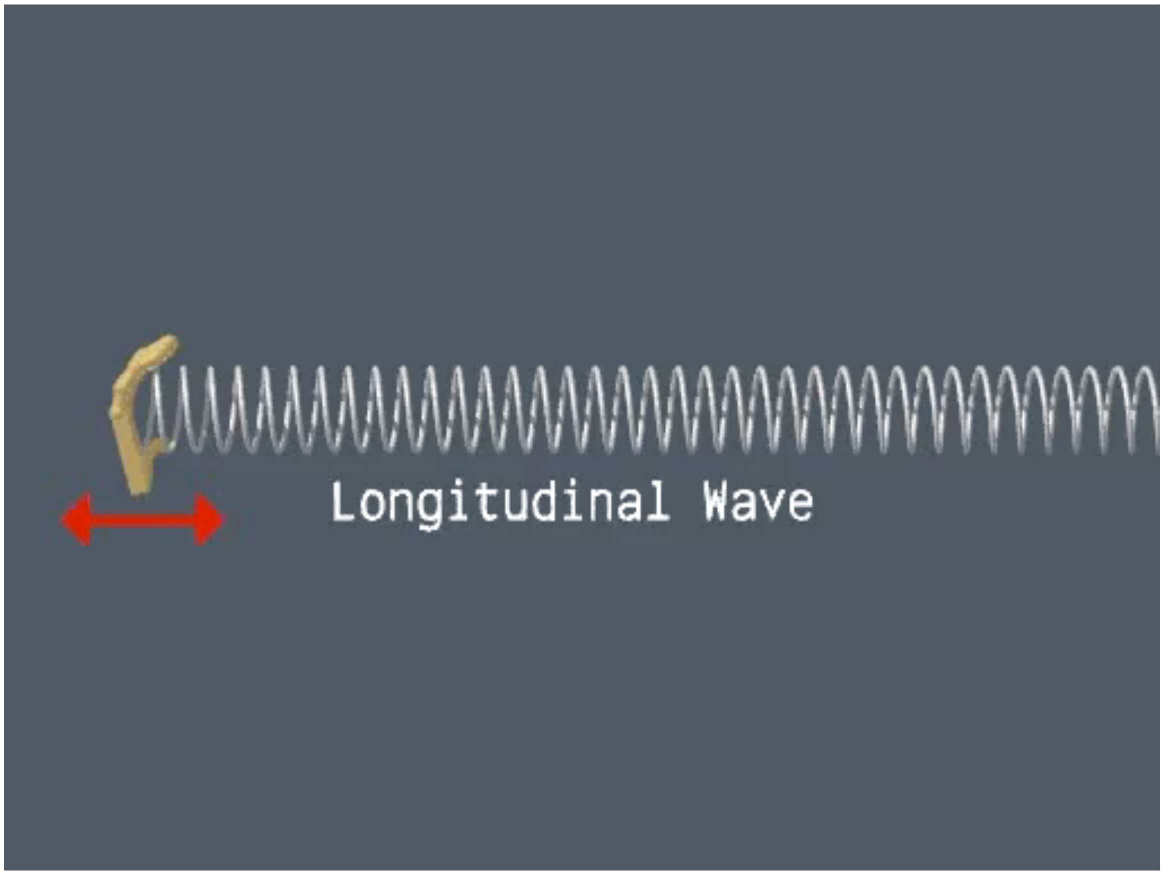
  - propagating waves, with the possibility of reflections.

- Other 1D waves include pressure waves (i.e. sound waves or seismic waves)

$$\frac{\partial^2 p}{\partial t^2} = c^2 \frac{\partial^2 p}{\partial x^2}$$

Waves/oscillations



Waves/oscillations

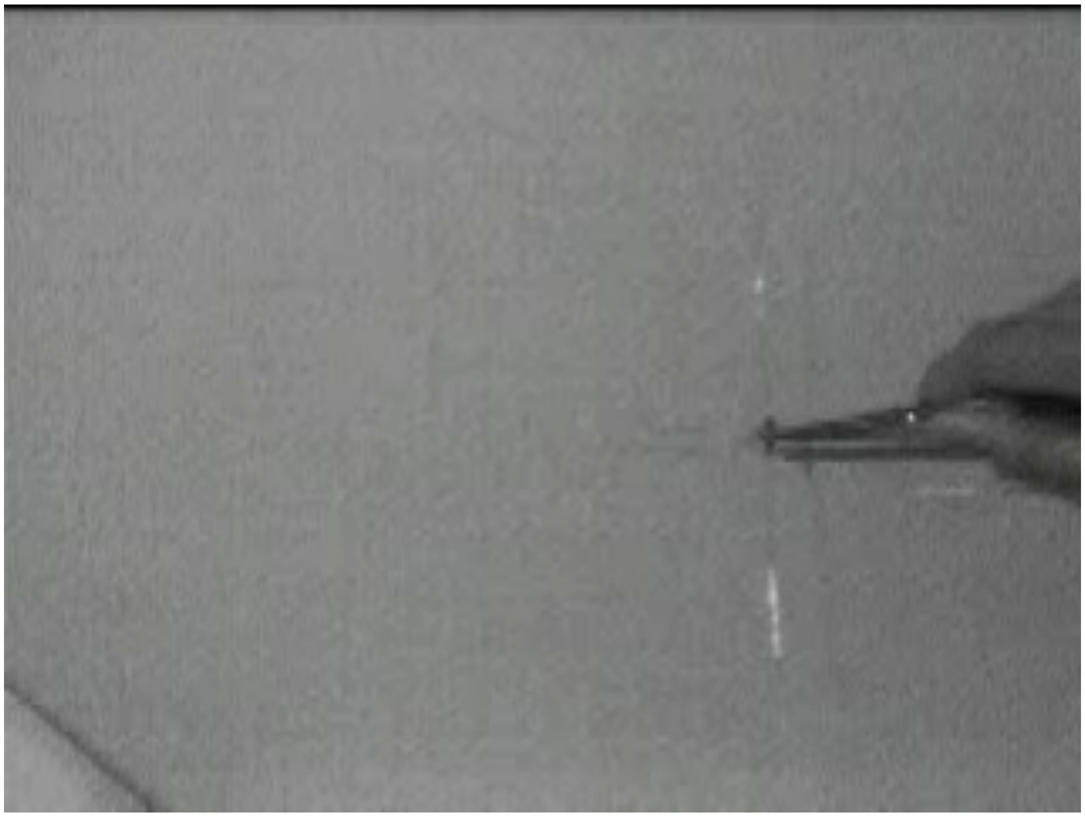
- 2D waves also have similar properties:
  - standing waves (oscillations) with different possible modes
  - propagating waves, with the possibility of reflections.

$$\frac{\partial^2 h}{\partial t^2} = c^2\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right)$$

- Examples:
  - Small-amplitude waves on the surface of water (gravity waves, capillary waves)
  - The vibrations of a surface under tension (e.g. guitar, drum, etc).

Waves/oscillations

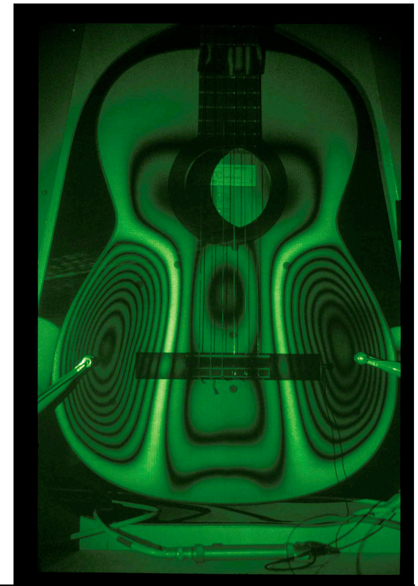
- Capillary waves

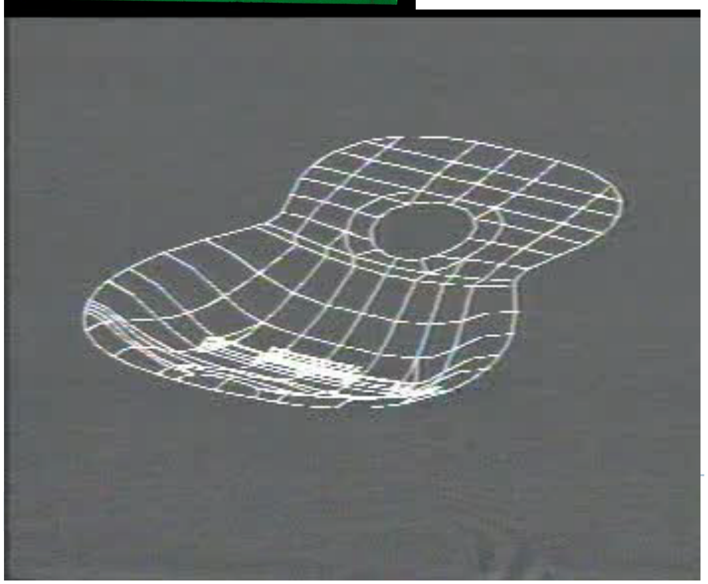


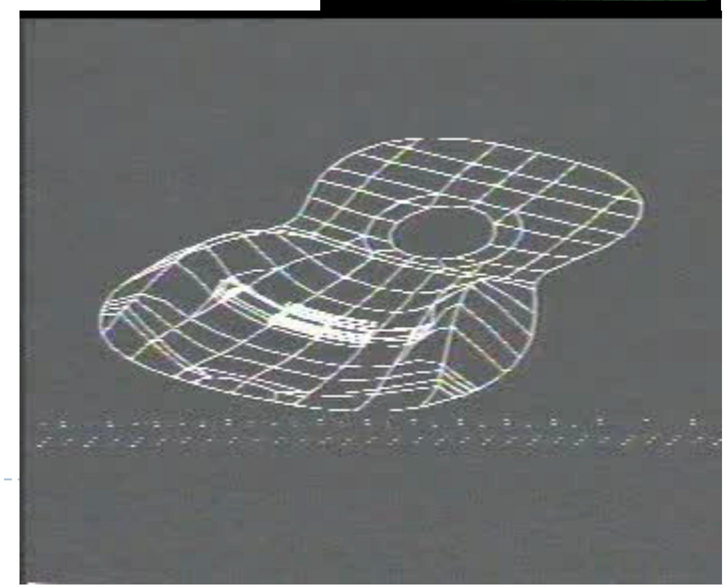
Waves/oscillations

Waves/oscillations

Standing waves in guitar excite propagating sound waves







Waves/oscillations

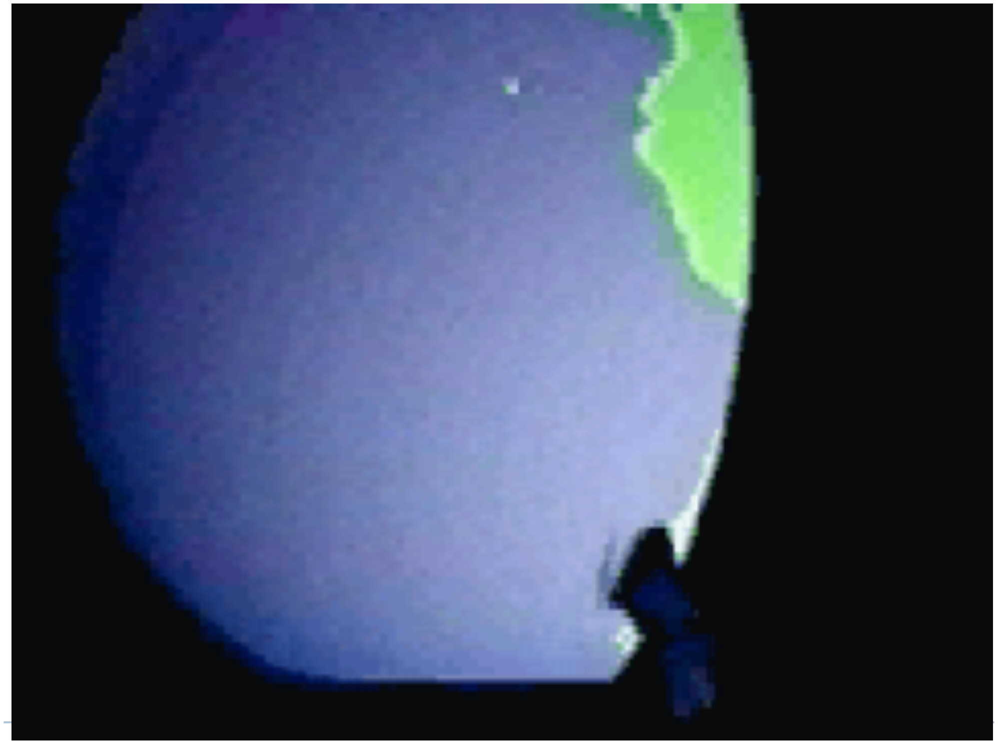
- Of course, waves in higher dimensions are also possible:
  - standing waves (oscillations) with different possible modes
  - propagating waves, with the possibility of reflections.

$$\frac{\partial^2 h}{\partial t^2} = c^2\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2}\right)$$

- Examples:
  - 3D seismic waves in the Earth, planets, stars…
  - Electromagnetic waves (light)
  - …

Waves/oscillations

- Note that some other types of waves are not modeled by such a simple wave equation.
  - Tsunami waves are perfect examples of weakly nonlinear waves



Waves/oscillations

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  - Waves breaking on the beach are very nonlinear – not well-understood at all actually, at least mathematically.

Waves/oscillations

- Note that some other types of waves are not modeled by such a simple wave equation.
  - Waves breaking on the beach are very nonlinear – not well-understood at all actually, at least mathematically.
  - Some surfers understand waves "intuitively" quite well…

Examples of PDEs.

2. The diffusion equation

The diffusion equation

- The diffusion equation

$$\frac{\partial f}{\partial t} = k\nabla^2 f$$

Diffusivity

- Note:
  - Note that the only difference with the wave equation is that the time-derivative is first-order!

- The most common example is the equation for the diffusion of heat (the heat equation):

$$\frac{\partial T}{\partial t} = k\nabla^2 T$$

The diffusion equation

- Because of this, we all have a very good intuition of what solutions of the diffusion equation are expected to do…
- Example:
  - Consider the 1D diffusion equation, with the following boundary condition and initial condition:

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2}$$

$$T(x,0) = 0 \qquad T(0,t) = 100 \quad T(L,t) = 0$$

  - You can guess the behavior of the solution quite easily by recasting this into a physical problem you're more familiar with…

The diffusion equation

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  - This is equivalent to a 1-D metal rod, initially at 0 degree, held at 0 degree at one end and at 100 degree at the other.

The diffusion equation

- Example:
  - Even if the analytical solution is not particularly transparent …

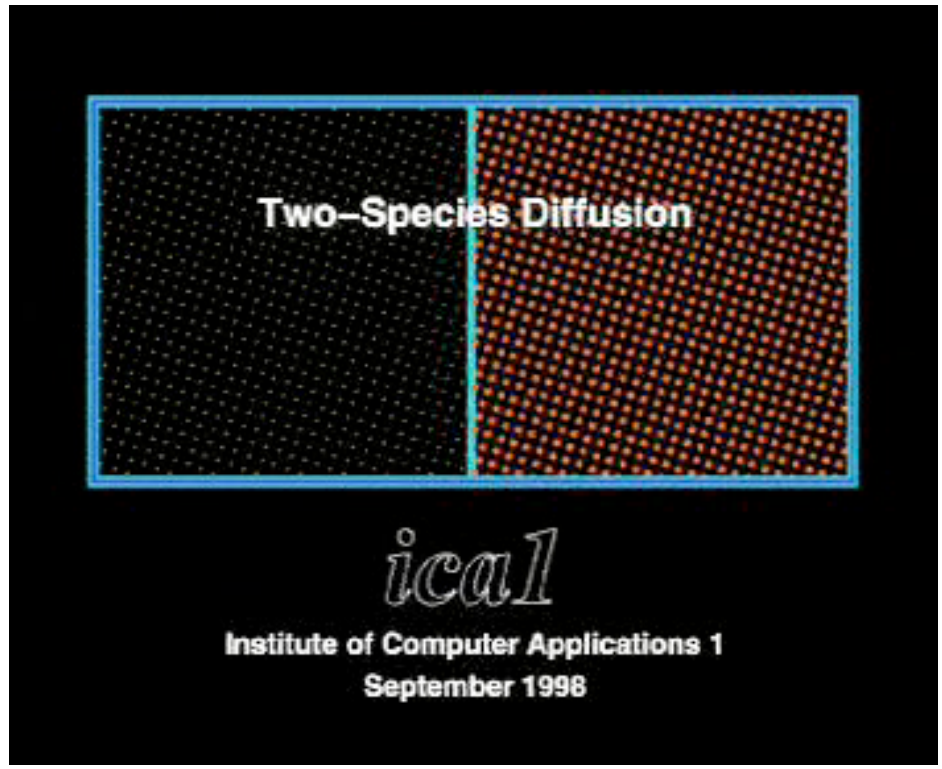
$$T(x,t) = 100\left(1 - \frac{x}{L}\right) - \sum_n \frac{200}{n\pi}\cos(n\pi)\sin\left(\frac{n\pi x}{L}\right)e^{-\frac{n^2\pi^2}{L^2}kt}$$

  - YOU still know what it should look like…
  - (IDL Movie)

The diffusion equation

- The diffusion equation is also a continuous way of describing the evolution of the probability distribution function of, say, dye molecules undergoing random walks caused by collisions with water molecules.
  - For a 2D Cartesian box,

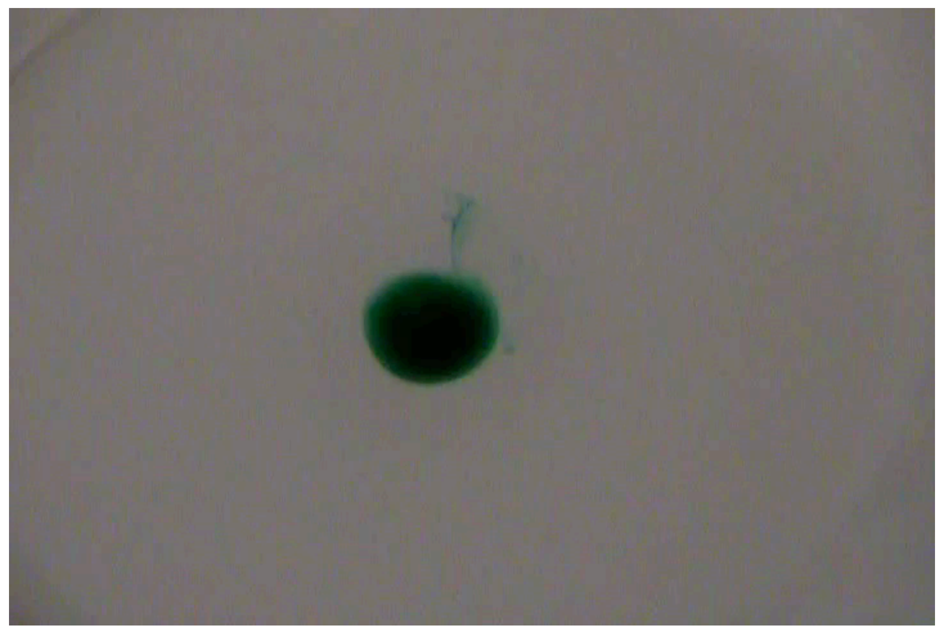
$$\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$



The diffusion equation

- The diffusion equation is also a continuous way of describing the evolution of the probability distribution function of, say, dye molecules undergoing random walks caused by collisions with water molecules.
  - For an axially symmetric patch of dye in water, for example,

$$\frac{\partial T}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$$



The diffusion equation

- Typical properties:
  - Diffusion "smoothes out" all sharp edges
  - The infinite-time evolution of the diffusion process is the smoothest possible function satisfying the boundary conditions.

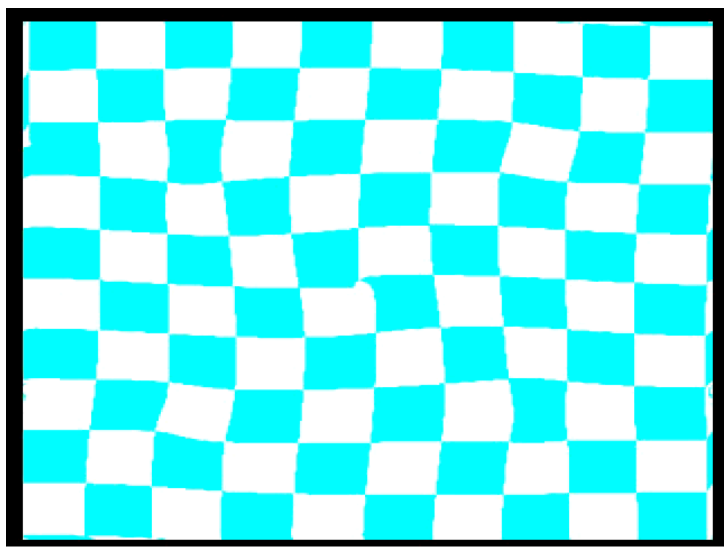
- Case of rod: linear function.
- Case of dye patch and 2-species diffusion in a finite-size box?
  - Constant functions.

(not) The diffusion equation

- Important note:
  - The diffusion must not be mixed up with the *transport equation:*

$$\frac{\partial f}{\partial t} = -\nabla \cdot (uf)$$

  - where u is the transport velocity field "transporting" the quantity f.



Diffusion vs transport.

- Important note:
  - The diffusion must not be mixed up with the *transport equation:*

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  - See how the transport equation *does not smooth out* the function f, it merely mixes it around.
  - In addition, the transport equation is *time-reversible*, while the diffusion equation is not…

Diffusion vs transport.

Examples of PDEs.

3. Laplace's equation

Laplace's equation

- Laplace's equation:

$$\nabla^2 f = 0$$

- Note:
  - This equation can be thought of as the "end-product" of a diffusion process, when the system has reached a steady state.

$$\frac{\partial f}{\partial t} = k\nabla^2 f \quad \text{when} \quad \frac{\partial}{\partial t} \to 0$$

  - So the solutions are always the "smoothest possible functions" which satisfy the boundary conditions…

Laplace's equation

- Laplace's equation on a square plate:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

  - with boundary conditions

$$f(0,y) = \sin(\pi y), f(1,y) = 0$$

$$f(x,0) = f(x,1) = 0$$

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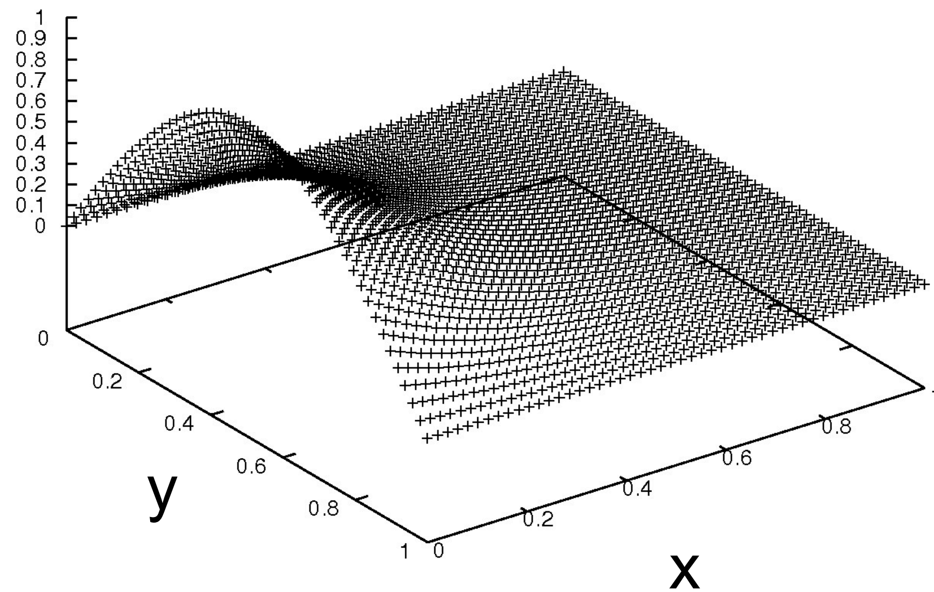

Laplace’s equation

- Thinking of Laplace’s equation as the “end-product” of a diffusion equation will also help you “guess” the solution..
- Example:
  - What is the solution of

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}=0$$

  - with boundary condition

$$f(1,\theta,\phi)=10$$

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  - with boundary condition

$$f(1,\theta,\phi)=10$$

  - Well, that's the Laplace equation on a sphere, i.e. the steady-state equilibrium of a sphere where the "temperature" f is held constant on the surface (r=1) and at f=10 for all times… so:

$$f(r,\theta,\phi)=10$$