After eliminating t we finally obtain

$$u(x, y) = 1 + \frac{n_0}{\sqrt{5}}(2x - y).$$

Notice that the parametric representation obtained in the current example is different from the one we derived in Example 2.16, since the parameter t we used here is half the parameter used in Example 2.16.

2.10 Exercises

2.1 Consider the equation $u_x + u_y = 1$, with the initial condition $u(x, 0) = f(x)$.
(a) What are the projections of the characteristic curves on the (x, y) plane?
(b) Solve the equation.

2.2 Solve the equation $xu_x + (x + y)u_y = 1$ with the initial conditions $u(1, y) = y$. Is the solution defined everywhere?

2.3 Let p be a real number. Consider the PDEs

$$xu_x + yu_y = pu \qquad -\infty < x < \infty, \quad -\infty < y < \infty.$$

(a) Find the characteristic curves for the equations.
(b) Let $p = 4$. Find an explicit solution that satisfies $u = 1$ on the circle $x^2 + y^2 = 1$.
(c) Let $p = 2$. Find two solutions that satisfy $u(x, 0) = x^2$, for every $x > 0$.
(d) Explain why the result in (c) does not contradict the existence–uniqueness theorem.

2.4 Consider the equation $yu_x - xu_y = 0 \ (y > 0)$. Check for each of the following initial conditions whether the problem is solvable. If it is solvable, find a solution. If it is not, explain why.

(a) $u(x, 0) = x^2$.
(b) $u(x, 0) = x$.
(c) $u(x, 0) = x, \quad x > 0$.

2.5 Let $u(x, y)$ be an integral surface of the equation

$$a(x, y)u_x + b(x, y)u_y + u = 0,$$

where $a(x, y)$ and $b(x, y)$ are positive differentiable functions in the entire plane. Define

$$D = \{(x, y), |x| < 1, |y| < 1\}.$$

(a) Prove that the projection on the (x, y) plane of each characteristic curve passing through a point in D intersects the boundary of D at exactly two points.
(b) Show that if u is positive on the boundary of D, then it is positive at every point in D.

(c) Suppose that u attains a local minimum (maximum) at a point $(x_0, y_0) \in D$. Evaluate $u(x_0, y_0)$.

(d) Denote by m the minimal value of u on the boundary of D. Assume $m > 0$. Show that $u(x, y) \geq m$ for all $(x, y) \in D$.

Remark This is an atypical example of a first-order PDE for which a maximum principle holds true. Maximum principles are important tools in the study of PDEs, and they are valid typically for second-order elliptic and parabolic PDEs (see Chapter 7).

2.6 The equation $xu_x + (x^2 + y)u_y + (y/x - x)u = 1$ is given along with the initial condition $u(1, y) = 0$.

(a) Solve the problem for $x > 0$. Compute $u(3, 6)$.

(b) Is the solution defined for the entire ray $x > 0$?

2.7 Solve the Cauchy problem $u_x + u_y = u^2, \quad u(x, 0) = 1$.

2.8 (a) Solve the equation $xuu_x + yuu_y = u^2 - 1$ for the ray $x > 0$ under the initial condition $u(x, x^2) = x^3$.

(b) Is there a unique solution for the Cauchy problem over the entire real line $-\infty < x < \infty$?

2.9 Consider the equation

$$uu_x + u_y = -\frac{1}{2}u.$$

(a) Show that there is a unique integral surface in a neighborhood of the curve

$$\Gamma_1 = \{(s, 0, \sin s) \mid -\infty < s < \infty\}.$$

(b) Find the parametric representation $x = x(t, s),\ y = y(t, s),\ u = u(t, s)$ of the integral surface S for initial condition of part (a).

(c) Find an integral surface S_1 of the same PDE passing through the initial curve

$$\Gamma_1 = \{(s, s, 0) \mid -\infty < s < \infty\}.$$

(d) Find a parametric representation of the intersection curves of the surfaces S and S_1.

Hint Try to characterize that curve relative to the PDE.

2.10 A river is defined by the domain

$$D = \{(x, y) \mid \ |y| < 1,\ -\infty < x < \infty\}.$$

A factory spills a contaminant into the river. The contaminant is further spread and convected by the flow in the river. The velocity field of the fluid in the river is only in the x direction. The concentration of the contaminant at a point (x, y) in the river and at time τ is denoted by $u(x, y, \tau)$. Conservation of matter and momentum implies that u satisfies the first-order PDE

$$u_\tau - (y^2 - 1)u_x = 0.$$

The initial condition is $u(x, y, 0) = e^y e^{-x^2}$.

(a) Find the concentration u for all (x, y, τ).

(b) A fish lives near the point $(x, y) = (2, 0)$ at the river. The fish can tolerate contaminant concentration levels up to 0.5. If the concentration exceeds this level, the fish will die at once. Will the fish survive? If yes, explain why. If no, find the time in which the fish will die.

Hint Notice that y appears in the PDE just as a parameter.

2.11 Solve the equation $(y^2 + u)u_x + yu_y = 0$ in the domain $y > 0$, under the initial condition $u = 0$ on the planar curve $x = y^2/2$.

2.12 Solve the equation $u_y + u^2 u_x = 0$ in the ray $x > 0$ under the initial condition $u(x, 0) = \sqrt{x}$. What is the domain of existence of the solution?

2.13 Consider the equation $uu_x + xu_y = 1$, with the initial condition $\left(\frac{1}{2}s^2 + 1, \frac{1}{6}s^3 + s, s\right)$. Find a solution. Are there other solutions? If not, explain why; if there *are* further solutions, find at least two of them, and explain the lack of uniqueness.

2.14 Consider the equation $xu_x + yu_y = 1/\cos u$.

(a) Find a solution to the equation that satisfies the condition $u(s^2, \sin s) = 0$ (you can write down the solution in the implicit form $F(x, y, u) = 0$).

(b) Find some domain of s values for which there exists a unique solution.

2.15 (a) Find a function $u(x, y)$ that solves the Cauchy problem

$$(x + y^2)u_x + yu_y + \left(\frac{x}{y} - y\right)u = 1\,, \quad u(x, 1) = 0 \quad x \in \mathbb{R}.$$

(b) Check whether the transversality condition holds.

(c) Draw the projections on the (x, y) plane of the initial condition and the characteristic curves emanating from the points $(2, 1, 0)$ and $(0, 1, 0)$.

(d) Is the solution you obtained in (a) defined at the origin $(x, y) = (0, 0)$? Explain your answer in light of the existence–uniqueness theorem.

2.16 Solve the Cauchy problem

$$xu_x + yu_y = -u, \quad u(\cos s, \sin s) = 1 \quad 0 \le s \le \pi.$$

Is the solution defined everywhere?

2.17 Consider the equation

$$xu_x + u_y = 1.$$

(a) Find a characteristic curve passing through the point $(1, 1, 1)$.

(b) Show that there exists a unique integral surface $u(x, y)$ satisfying $u(x, 0) = \sin x$.

(c) Is the solution defined for all x and y?

2.18 Consider the equation $uu_x + u_y = -\frac{1}{2}u$.

(a) Find a solution satisfying $u(x, 2x) = x^2$.

(b) Is the solution unique?

2.19 (a) Find a function $u(x, y)$ that solves the Cauchy problem

$$x^2 u_x + y^2 u_y = u^2, \quad u(x, 2x) = x^2 \quad x \in \mathbb{R}.$$

(b) Check whether the transversality condition holds.

(c) Draw the projections on the (x, y) plane of the initial curve and the characteristic curves that start at the points $(1, 2, 1)$ and $(0, 0, 0)$.
(d) Is the solution you found in part (a) defined for all x and y?

2.20 Consider the equation

$$yu_x - uu_y = x.$$

(a) Write a parametric representation of the characteristic curves.
(b) Solve the Cauchy problem

$$yu_x - uu_y = x, \qquad u(s, s) = -2s \qquad -\infty < s < \infty.$$

(c) Is the following Cauchy problem solvable:

$$yu_x - uu_y = x, \qquad u(s, s) = s \qquad -\infty < s < \infty?$$

(d) Set

$$w_1 = x + y + u, \quad w_2 = x^2 + y^2 + u^2, \quad w_3 = xy + xu + yu.$$

Show that $w_1(w_2 - w_3)$ is constant along each characteristic curve.

2.21 (a) Find a function $u(x, y)$ that solves the Cauchy problem

$$xu_x - yu_y = u + xy, \quad u(x, x) = x^2 \quad 1 \le x \le 2.$$

(b) Check whether the transversality condition holds.
(c) Draw the projections on the (x, y) plane of the initial curve and the characteristic curves emanating from the points $(1, 1, 1)$ and $(2, 2, 4)$.
(d) Is the solution you found in (a) well defined in the entire plane?

2.22 Solve the Cauchy problem $u_x^2 + u_y = 0$, $u(x, 0) = x$.

2.23 Let $u(x, t)$ be the solution to the Cauchy problem

$$u_t + cu_x + u^2 = 0, \quad u(x, 0) = x,$$

where c is a constant, t denotes time, and x denotes a space coordinate.
(a) Solve the problem.
(b) A person leaves the point x_0 at time $t = 0$, and moves in the positive x direction with a velocity c (i.e. the quantity $x - ct$ is fixed for him). Show that if $x_0 > 0$, then the solution as seen by the person approaches zero as $t \to \infty$.
(c) What will be observed by such a person if $x_0 < 0$, or if $x_0 = 0$?

2.24 (a) Solve the problem

$$xu_x - uu_y = y, \qquad u(1, y) = y \qquad -\infty < y < \infty.$$

(b) Is the solution unique? What is the maximal domain where it is defined?

2.25 Find at least five solutions for the Cauchy problem

$$u_x + u_y = 1, \quad u(x, x) = x.$$

2.26 (a) Solve the problem

$$\begin{aligned} xu_y - yu_x + u &= 0, \\ u(x, 0) &= 1 \qquad x > 0. \end{aligned}$$

(b) Is the solution unique? What is the maximal domain where it is defined?

2.27 (a) Use the Lagrange method to find a function $u(x, y)$ that solves the problem

$$uu_x + u_y = 1 \tag{2.100}$$

$$u(3x, 0) = -x \quad -\infty < x < \infty. \tag{2.101}$$

(b) Show that the curve $\{(3x, 2, 4 - 3x) | -\infty < x < \infty\}$ is contained in the solution surface $u(x, y)$.
(c) Solve

$$\begin{aligned} uu_x + u_y &= 1 \\ u(3x, 2) &= 4 - 3x \quad -\infty < x < \infty. \end{aligned}$$

2.28 Analyze the following problems using the Lagrange method. For each problem determine whether there exists a unique solution, infinitely many solutions or no solution at all. If there is a unique solution, find it; if there are infinitely many solutions, find at least two of them. Present all solutions explicitly.
(a)

$$\begin{aligned} xuu_x + yuu_y &= x^2 + y^2 \quad x > 0,\ y > 0, \\ u(x, 1) &= \sqrt{x^2 + 1}. \end{aligned}$$

(b)

$$\begin{aligned} xuu_x + yuu_y &= x^2 + y^2 \quad x > 0,\ y > 0, \\ u(x, x) &= \sqrt{2}x. \end{aligned}$$

2.29 Consider the equation

$$xu_x + (1 + y)u_y = x(1 + y) + xu.$$

(a) Find the general solution.
(b) Assume an initial condition of the form $u(x, 6x - 1) = \phi(x)$. Find a necessary and sufficient condition for ϕ that guarantees the existence of a solution to the problem. Solve the problem for the appropriate ϕ that you found.
(c) Assume an initial condition of the form $u(-1, y) = \psi(y)$. Find a necessary and sufficient condition for ψ that guarantees the existence of a solution to the problem. Solve the problem for the appropriate ψ that you found.

(d) Explain the differences between (b) and (c).

2.30 (a) Find a compatibility condition for the Cauchy problem

$$u_x^2 + u_y^2 = 1, \qquad u(\cos s, \sin s) = 0 \quad 0 \le s \le 2\pi.$$

(b) Solve the above Cauchy problem.

(c) Is the solution uniquely defined?