Take Home Midterm 2013

Problem 1:

A 1D advection-diffusion equation is of the form

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = k\frac{\partial^2 u}{\partial x^2}$$

(a) Solve this equation on the interval $x \in [-10, 10]$ with initial and boundary conditions:

$$u(-10,t) = u(10,t) = 0$$
$$u(x,0) = e^{-x^2/2} - e^{-50}$$

(b) Plot the solution for $t = 0, 1, 2, 3, 4, 5$ for the 3 following cases:

- $v = 0$, $k = 1$
- $v = 1$, $k = 0$
- $v = 1$, $k = 1$

(c) Search online or in textbooks for the term "advection-diffusion" and describe in words how the behavior of your solutions fits with the online descriptions you found. What kind of physical problem could this correspond to?

Problem 2:

Your instructor is currently writing a paper (Oglethorpe, Garaud and Wood, in prep) in which she has to study the following problem. On the interval $z \in (0, H)$, we have

$$\frac{\partial v}{\partial t} + \left(\frac{2\Omega(t)R}{N\lambda}\right)^2 \kappa\frac{\partial^4 v}{\partial z^4} - \left(\frac{2\Omega(t)}{N}\right)^2 \kappa\frac{\partial^2 v}{\partial z^2} = 0 \tag{1}$$

where $\Omega(t) = \Omega_0 \left(\frac{t}{t_0}\right)^{-1/2}$, and where N, λ, R, κ, Ω_0 and t_0 are just constants. The boundary conditions on $v(z,t)$ are:

$$\frac{\partial v}{\partial z}(0,t) = 0 \text{ and } \frac{\partial^3 v}{\partial z^3}(0,t) = 0,$$
$$\frac{\partial v}{\partial z}(H,t) = 0 \text{ and } v(H,t) = K\frac{\partial^3 v}{\partial z^3}(H,t),$$

where K is a positive constant. The initial conditions are arbitrary, so that

$$v(z,t_0) = v_0(z).$$

The paper contains the following statements. For each point marked (**N**) where $N = 1..10$, prove the corresponding statement and/or perform the required calculation.

"To solve Equation (1), we note that it is separable in z and t, and write

$$v(z,t) = \sum_n V_n(t) Z_n(z),$$

where the vertical modes $Z_n(z)$ satisfy

$$\mathcal{L}(Z_n) \equiv \frac{d^4 Z_n}{dz^4} - \frac{\lambda^2}{R^2}\frac{d^2 Z_n}{dz^2} = \left(\frac{\mu_n}{R^2}\right)^2 Z_n \quad \text{(**1**)} \tag{2}$$

*We see that the boundary conditions on the eigenfunctions are homogeneous (**2**). It can easily be shown that the operator $\mathcal{L}$ on the left-hand-side of (2) is symmetric (**3**) with these boundary conditions, which then implies that the vertical eigenmodes are orthogonal, with*

$$\int_0^H Z_n(z) Z_m(z) dz = \delta_{nm} \int_0^H Z_n^2(z) dz. \quad \text{(**4**)}$$

*It can also be shown by considering the integral $\int_0^H Z_n \mathcal{L}(Z_n) dz$, suitably integrating it by parts, and applying the boundary conditions, that the eigenvalues associated with the operator $\mathcal{L}$ and our boundary conditions must be strictly positive (**5**), hence our choice of writing them as $(\mu_n/R^2)^2$ in Equation (2).*

Since (2) is an equation with constant coefficients, we seek solutions of the form $e^{\sigma_n z}$ and find four solutions for σ_n: $\pm\sigma_{1,n}$ and $\pm i\sigma_{2,n}$, where

$$\sigma_{1,n} = \frac{1}{R}\left[\sqrt{\mu_n^2 + \frac{\lambda^4}{4}} + \frac{\lambda^2}{2}\right]^{1/2}, \ \sigma_{2,n} = \frac{1}{R}\left[\sqrt{\mu_n^2 + \frac{\lambda^4}{4}} - \frac{\lambda^2}{2}\right]^{1/2}. \quad \text{(**6**)}$$

Applying the boundary conditions we then have

$$Z_n(z) = \frac{\sigma_{2,n}}{\sigma_{1,n}} \frac{\sin(\sigma_{2,n} H)}{\sinh(\sigma_{1,n} H)} \cosh\left(\sigma_{1,n}(z)\right) + \cos\left(\sigma_{2,n}(z)\right), \quad \text{(**7**)}$$

*while the μ_n coefficients satisfy: (**8** : write the equation that the μ_n coefficients satisfy).*

*Projecting (1) onto the Z_n functions gives (**9**: prove the equation below and write what is $\tau_n(t)$ in terms of all the other parameters introduced)*

$$\frac{dV_n}{dt} + \frac{V_n}{\tau_n(t)} = 0,$$

*Solving this equation, using the solution to reconstruct $v(z,t)$, and applying initial conditions finally yields (**10**: write the complete formal solution). "*

Problem 3:

What are the eigenfrequencies of a rectangular drum of size $H \times L$, assuming that the wave-speed of the drum is c? Justify all the steps of your calculation very carefully.