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MIDTERM 2013

Problem 1

$$\begin{cases} \frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = k\frac{\partial^2 u}{\partial x^2} \\ u(-10,t) = u(10,t) = 0 \\ u(x,0) = e^{-\frac{x^2}{2}} - e^{-\frac{100}{2}} = e^{-\frac{x^2}{2}} - e^{-50} \end{cases}$$

(a) $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - v\frac{\partial u}{\partial x}$ if $u(x,t) = A(x)B(t)$

So
$$\begin{cases} \frac{dB}{dt} = -\lambda B & (1) \\ k\frac{d^2A}{dx^2} - v\frac{dA}{dx} = -\lambda A & (2) \end{cases}$$
10 up to here

From (2) we get: $\frac{d^2A}{dx^2} - \frac{v}{k}\frac{dA}{dx} + \frac{\lambda}{k}A = 0$

$\lambda \neq 0$ Seeking solutions of the form $A = e^{rx}$

$$r^2 - \frac{v}{k}r + \frac{\lambda}{k} = 0$$

$$\Rightarrow r = \frac{\frac{v}{k} \pm \sqrt{\frac{v^2}{k^2} - 4\frac{\lambda}{k}}}{2} = \frac{v}{2k} \pm \sqrt{\frac{v^2}{4k^2} - \frac{\lambda}{k}}$$
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- If r is real, then we have 2 exponential solutions
  → this can't be matched to the solutions.
- If r is complex, with
$$r = \frac{v}{2k} \pm i\sqrt{\frac{\lambda}{k} - \frac{v^2}{4k^2}}$$

then $A(x) = e^{\frac{v}{2k}x}\left\{a\cos\left(\sqrt{\frac{\lambda}{k} - \frac{v^2}{4k^2}}\,x\right) + b\sin\left(\sqrt{\frac{\lambda}{k} - \frac{v^2}{4k^2}}\,x\right)\right\}$

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To satisfy the bcs we need:

$$\begin{cases} a\cos\left(10\sqrt{\frac{\lambda}{k}-\frac{V^2}{4k^2}}\right) - b\sin\left(10\sqrt{\frac{\lambda}{k}-\frac{V^2}{4k^2}}\right) = 0 & (1) \\ a\cos\left(10\sqrt{\frac{\lambda}{k}-\frac{V^2}{4k^2}}\right) + b\sin\left(10\sqrt{\frac{\lambda}{k}-\frac{V^2}{4k^2}}\right) = 0 & (2) \end{cases}$$

This system only has solutions if the determinant is 0:

$$2\cos\left(10\sqrt{\frac{\lambda}{k}-\frac{V^2}{4k^2}}\right)\sin\left(10\sqrt{\frac{\lambda}{k}-\frac{V^2}{4k^2}}\right) = 0$$

so if $\quad \sin\left(20\sqrt{\frac{\lambda}{k}-\frac{V^2}{4k^2}}\right) = 0$

$$\Rightarrow \quad 20\sqrt{\frac{\lambda}{k}-\frac{V^2}{4k^2}} = n\pi$$

$$\Rightarrow \quad \frac{\lambda}{k}-\frac{V^2}{4k^2} = \left(\frac{n\pi}{20}\right)^2 \Rightarrow \lambda = \left(\frac{V^2}{4k^2}+\frac{n^2\pi^2}{400}\right)k$$

20.

Also, from (1) we need

$$a_n\cos\left(\frac{n\pi}{2}\right) = b_n\sin\left(\frac{n\pi}{2}\right)$$

So if n is odd, $b_n = 0$

n is even, $a_n = 0$

Finally we get:

$$A_n(x) = \begin{cases} e^{\frac{v}{2k}x}\sin\left(\frac{n\pi x}{20}\right) & \text{if } n \text{ is even} \\ e^{\frac{v}{2k}x}\cos\left(\frac{n\pi x}{20}\right) & \text{if } n \text{ is odd.} \end{cases}$$

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and $\quad \lambda_n = \left(\frac{V^2}{4k^2}+\frac{n^2\pi^2}{400}\right)k$

So:

$$u(x,t) = \sum_{n=\text{odd}} a_n e^{\frac{vx}{2k}}\cos\left(\frac{n\pi x}{20}\right)e^{-\lambda_n t} + \sum_{n\,\text{even}} b_n e^{\frac{vx}{2k}}\sin\left(\frac{n\pi x}{20}\right)e^{-\lambda_n t}$$

(Note: if $\lambda = 0$ then $\frac{d^2A}{dx^2} = -\frac{V}{k}\frac{dA}{dx} \Rightarrow \frac{dA}{dx} = \tilde{a}e^{-\frac{V}{k}x}$

$A(x) = ae^{-\frac{V}{k}x} + b \rightarrow$ can't fit the bcs)

At $t=0$:

$$e^{-\frac{x^2}{2}} - e^{-\frac{100}{2}} = \sum_{n\,odd}' e^{\frac{v}{2k}x} a_n \cos\left(\frac{n\pi x}{20}\right) + \sum_{n\,even}' e^{\frac{v}{2k}x} c_n \sin\left(\frac{n\pi x}{20}\right)$$

The orthogonality for this case is given by the Sturm-Liouville form:

$$\frac{d}{dx}\left(e^{-\frac{v}{k}x}\frac{dA}{dx}\right) = -e^{-\frac{v}{k}x}\frac{\lambda}{k}A$$

so $r(x) = \frac{1}{k}e^{-\frac{v}{k}x}$ and $p(x) = e^{-\frac{v}{k}x}$

$$\Rightarrow \langle A_n, A_m\rangle = \int_{-10}^{10} \frac{1}{k} e^{-\frac{v}{k}x} A_n(x) A_m(x)\,dx$$

<u>Check</u>

Suppose n is odd:

- if m is even then

$$\int_{-10}^{10} \frac{1}{k} \cos\left(\frac{n\pi x}{20}\right) \sin\left(\frac{m\pi x}{20}\right) dx$$

$$= \int_{-10}^{10} \frac{1}{2k} \sin\left(\frac{n\pi x}{10}\right) dx = 0$$

↑ symmetric interval ↑ odd function

- if m is odd too then

$$\int_{-10}^{10} \frac{1}{k} \cos\left(\frac{n\pi x}{20}\right) \cos\left(\frac{m\pi x}{20}\right) dx = 0 \quad \text{too if } n \neq m$$

(etc...)

If $n = m =$ odd

$$\int_{-10}^{10} \frac{1}{k} \cos^2\left(\frac{n\pi x}{20}\right) dx = \int_{-10}^{10} \frac{1}{2k}\left(1 - \cos\left(\frac{n\pi x}{10}\right)\right) dx$$

$$= \frac{10}{k}$$ (and similarly for $n = m =$ even)

So : projecting the IC onto the Ans we get

n odd:

$$\int_{-10}^{10} \frac{1}{k} e^{-\frac{v}{k}x} e^{\frac{v}{2k}x} \cos\left(\frac{n\pi x}{20}\right)\left(e^{-\frac{x^2}{2}} - e^{-50}\right) dx = \frac{10 a_n}{k}$$

So

$$a_n = \frac{1}{10}\int_{-10}^{10} e^{-\frac{v}{2k}x} \cos\left(\frac{n\pi x}{20}\right)\left(e^{-\frac{x^2}{2}} - e^{-50}\right) dx$$

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n even:

$$\int_{-10}^{10} \frac{1}{k} e^{-\frac{v}{k}x} e^{\frac{v}{2k}x} \sin\left(\frac{n\pi x}{20}\right)\left(e^{-\frac{x^2}{2}} - e^{-50}\right) dx = \frac{10 b_n}{k}$$

So

$$b_n = \frac{1}{10}\int_{-10}^{10} e^{-\frac{v}{2k}x} \sin\left(\frac{n\pi x}{20}\right)\left(e^{-\frac{x^2}{2}} - e^{-50}\right) dx$$

→ This completes the solution 10

(each integral can be written in terms of rather ugly erf if desired).

<u>Case $v=0, k=1$</u>

Then $$a_n = \frac{1}{10}\int_{-10}^{10} \cos\left(\frac{n\pi x}{20}\right)\left(e^{-\frac{x^2}{2}} - e^{-50}\right) dx$$

$$b_n = \frac{1}{10}\int_{-10}^{10} \sin\left(\frac{n\pi x}{20}\right)\left(e^{-\frac{x^2}{2}-50}\right) dx$$

20 for plots

or 10/10 plots + discuss

<u>Case $v=1 \quad k=\epsilon$</u> (Note: $k=0$ can't be taken directly because the solution blows up)

Then $$a_n = \frac{1}{10}\int_{-10}^{10} e^{\frac{-1}{2\epsilon}x} \cos\left(\frac{n\pi x}{20}\right)\left(e^{-\frac{x^2}{2}} - e^{-50}\right) dx$$

$$b_n = \frac{1}{10}\int_{-10}^{10} e^{\frac{-1}{2\epsilon}x} \sin\left(\frac{n\pi x}{20}\right)\left(e^{-\frac{x^2}{2}} - e^{-50}\right) dx$$

y

Problem 2 10/question

(1) If we separate variables into $V(t)Z(z)$ we get

$$Z(z)\frac{dV}{dt} + V(t).\left(\frac{2\Omega(t)}{N}\right)^2 k\left\{\left(\frac{R}{\lambda}\right)^2\frac{d^4Z}{dz^4} - \frac{d^2Z}{dz^2}\right\} = 0$$

$$\Rightarrow \left(\frac{N}{2\Omega(t)}\right)^2 \cdot \frac{1}{k}\left(\frac{\lambda}{R}\right)^2 \frac{1}{V}\frac{dV}{dt} = -\frac{1}{Z}\left\{\frac{d^4Z}{dz^4} - \frac{d^2Z}{dz^2}\left(\frac{\lambda}{R}\right)^2\right\} = \text{constant}$$

If this constant is $-\left(\frac{\mu}{R^2}\right)^2$ then we have, as required

$$\frac{d^4Z}{dz^4} - \left(\frac{\lambda}{R}\right)^2\frac{d^2Z}{dz^2} = \left(\frac{\mu}{R^2}\right)^2 Z$$

Supposing that there are many such constants, each indexed with n, we recover (2)

(2) The boundary conditions on $V(z,t)$ are

$$\left.\frac{\partial V}{\partial z}\right|_{z=0} = 0 \Rightarrow \frac{\partial}{\partial z}\sum_n V_n(t)Z_n(0) = 0 \Rightarrow \frac{dZ_n}{dz} = 0$$

(similarly at $z = H$) for all times

$$\left.\frac{\partial^3 V}{\partial z^3}\right|_{z=0} = 0 \Rightarrow \frac{\partial^3}{\partial z^3}\sum_n V_n(t)Z_n(z) = 0 \Rightarrow \frac{d^3Z_n}{dz^3} = 0$$

for all times

$$V = K\frac{\partial^3 V}{\partial z^3} = 0 \Rightarrow \sum_n V_n(t)\left(Z_n(z) - K\frac{d^3Z_n}{dz^3}\right) = 0 \quad \text{for all times}$$

So $Z_n(z) = K\frac{d^3Z_n}{dz^3}$

→ all these BCs are homogeneous. (i.e., they would be satisfied by the trivial solution).

(3) To prove symmetry, consider

$$\int_0^H Z_n\left(\frac{d^4Z_m}{dz^4}-\frac{\lambda^2}{R^2}\frac{d^2Z_m}{dz^2}\right)dz$$

$$=\left[Z_n\frac{d^3Z_m}{dz^3}\right]_0^H-\int_0^H\frac{dZ_n}{dz}\frac{d^3Z_m}{dz^3}-\frac{\lambda^2}{R^2}\cancel{\left[Z_n\frac{dZ_m}{dz}\right]_0^H}^{\,0}+\int_0^H\frac{dZ_n}{dz}\frac{dZ_m}{dz}\,dz$$

$$=Z_n(H)\cdot\frac{1}{K}Z_m(H)-0-\cancel{\left[\frac{dZ_n}{dz}\frac{d^2Z_m}{dz^2}\right]_0^H}^{\,0}+\int_0^H\frac{d^2Z_n}{dz^2}\frac{d^2Z_m}{dz^2}\,dz+\int_0^H\frac{dZ_n}{dz}\frac{dZ_m}{dz}\,dz$$

$$=\int_0^H Z_m\left(\frac{d^4Z_n}{dz^4}-\frac{\lambda^2}{R^2}\frac{d^2Z_n}{dz^2}\right)dz$$ since each term remaining above is invariant under permutation of n and m.

(4) $$\int_0^H Z_n(z)Z_m(z)\,dz=\int_0^H\frac{Z_n(z)}{\left(\frac{\mu_m}{R^2}\right)^2}\mathcal{L}(Z_m(z))\,dz$$ by $\mathcal{L}(Z_m)=\left(\frac{\mu_m}{R^2}\right)^2 Z_m$

$$=\int_0^H\frac{Z_m(z)\,\mathcal{L}(Z_n)}{\left(\frac{\mu_m}{R^2}\right)^2}\,dz$$ by symmetry

$$=\int_0^H\frac{Z_m\,\mathcal{L}(Z_n)}{\left(\frac{\mu_n}{R^2}\right)^2}\,dz$$ by $\mathcal{L}(Z_n)=\left(\frac{\mu_n}{R^2}\right)^2 Z_n$

$\rightarrow$ so this is only true if $\mu_n=\mu_m$ or if

$\int_0^H Z_nZ_m\,dz=0$ for $\mu_n\neq\mu_m$. $\rightarrow$ orthogonality is proved.

(5) $$\int_0^H Z_n\,\mathcal{L}(Z_n)\,dz=\int_0^H Z_n\left[\frac{d^4Z_n}{dz^4}-\frac{\lambda^2}{R^2}\frac{d^2Z_n}{dz^2}\right]dz$$

$$=\frac{1}{K}Z_n^2+\int_0^H\left(\frac{d^2Z_n}{dz^2}\right)^2+\left(\frac{dZ_n}{dz}\right)^2 dz$$ (using result of (3))

≥ 0 and in fact, >0 as long as Z_n is not $\equiv 0$

This means that if $\mathcal{L}(Z_n)=$ constant $\cdot Z_n$ that constant must be >0.

(6) $$\sigma_n^4 - \left(\frac{\lambda}{R}\right)^2 \sigma_n^2 - \left(\frac{\mu_n}{R^2}\right)^2 = 0$$

$$\Rightarrow \quad \sigma_n^2 = \frac{\left(\frac{\lambda}{R}\right)^2 \pm \sqrt{\left(\frac{\lambda}{R}\right)^4 + 4\left(\frac{\mu_n}{R^2}\right)^2}}{2}$$

$$\sigma_n^2 = \frac{1}{R^2}\left[\frac{\lambda^2}{2} \pm \sqrt{\frac{1}{4}\lambda^4 + \mu^2}\right]$$

The + solution is > 0 so this leads to $\pm\sigma_{1,n}$ solutions with

$$\sigma_{1,n} = \frac{1}{R}\left[\frac{\lambda^2}{2} + \sqrt{\frac{\lambda^4}{4} + \mu^2}\right]^{1/2}$$

The − solution is < 0 so ---- $\pm i\sigma_{2,n}$ solutions with

$$\sigma_{2,n} = \frac{1}{R}\left[-\frac{\lambda^2}{2} + \sqrt{\frac{\lambda^4}{4} + \mu^2}\right]$$

(7) The full solutions are therefore

$$Z_n(z) = a_1 \sin(\sigma_{2n} z) + a_2 \cos(\sigma_{2n} z) + a_3 \sinh(\sigma_{1n} z) + a_4 \cosh(\sigma_{1n} z)$$

For $\frac{dZ_n}{dz} = 0$ at $z = 0$ we have

$$+a_1 \sigma_{2n} + a_3 \sigma_{1n} = 0$$

$$\frac{d^3 Z_n}{dz^3} = 0 \quad \text{at } z = 0$$

$$-a_1 \sigma_{2n}^3 + a_3 \sigma_{1n}^3 = 0$$

$\Rightarrow$ this only has one solution, $a_1 = a_3 = 0$

For $\frac{dZ_n}{dz} = 0$ at $z = H$

$$-\sigma_{2n} a_2 \sin(\sigma_{2n} H) + \sigma_{1n} a_4 \sinh(\sigma_{1n} H) = 0$$

then from the last BC:

$$a_2 \cos(\sigma_{2n} H) + a_4 \cosh(\sigma_{1n} H) = K\left[\sin(\sigma_{2n} H)\, \sigma_{2n}^3 a_2 + a_4 \sigma_{1n}^3 \sinh(\sigma_{1n} H\right.$$

So: $$a_4 = \frac{\sigma_{2n}}{\sigma_{1n}} \frac{\sin(\sigma_{2n} H)}{\sinh(\sigma_{1n} H)} a_2$$

$$\Rightarrow Z_n(z) = \frac{\sigma_{2n}}{\sigma_{1n}} \frac{\sin(\sigma_{2n} H)}{\sinh(\sigma_{1n} H)} \cosh(\sigma_{1n} z) + \cos(\sigma_{2n} z) \quad \text{as required}$$

(8) From the last BC:

$$\cancel{a_2}\cos(\sigma_{2n}H) + \frac{\sigma_{2n}}{\sigma_{1n}}\frac{\sin(\sigma_{2n}H)}{\sinh(\sigma_{1n}H)}\cancel{a_2}$$

$$= K\left[\sin(\sigma_{2n}H)\sigma_{2n}^3 \cancel{a_2} + \frac{\sigma_{2n}}{\sigma_{1n}}\frac{\sin(\sigma_{2n}H)}{\cancel{\sinh(\sigma_{1n}H)}}\cancel{a_2}\,\sigma_{1n}^3\,\cancel{\sinh(\sigma_{1n}H)}\right]$$

$$\Rightarrow \frac{1}{\tan(\sigma_{2n}H)} + \frac{\sigma_{2n}}{\sigma_{1n}}\frac{1}{\sinh(\sigma_{1n}H)}$$

$$= K\left[\sigma_{2n}^3 + \frac{\sigma_{2n}}{\sigma_{1n}}\sigma_{1n}^3\right] = \sigma_{2n}K\left[\sigma_{2n}^2 + \sigma_{1n}^2\right]$$

(9) Let's $\int_0^H (1)\cdot Z_n(z)\,dz$

$$\mathcal{L}(Z_m) = \left(\frac{\mu_m}{R^2}\right)^2 Z_m$$

$$\int_0^H Z_n(z)\frac{\partial}{\partial t}\sum_m{}' Z_m(z)V_m(t)\,dz + \left(\frac{2\Omega(t)R}{N\lambda}\right)^2 k\int_0^H Z_n(z)\sum_m\underbrace{\left(\frac{d^4Z_m}{dz^4} - \frac{\lambda^2}{R^2}\frac{d^2Z_m}{dz^2}\right)}_{}V_m(t)\,dz$$

$$\|Z_n\|^2\dot{V}_n(t) + \left(\frac{2\Omega(t)R}{N\lambda}\right)^2 k\sum\int_0^H\left(\frac{\mu_m}{R^2}\right)^2 Z_n(z)Z_m(z)V_m(t)\,dz = 0$$

$$\Rightarrow \cancel{\|Z_n\|^2}\dot{V}_n(t) + \left(\frac{2\Omega(t)R}{N\lambda}\right)^2 k\left(\frac{\mu_m}{R^2}\right)^2 V_m(t)\cancel{\|Z_n\|^2} = 0$$

$$\text{so } \dot{V}_n(t) + \frac{V_m(t)}{\tau_n(t)} = 0$$

$$\Rightarrow \tau_n^{-1}(t) = \left(\frac{2\Omega(t)R}{N\lambda}\right)^2 k\left(\frac{\mu_m}{R^2}\right)^2$$

$$\tau_n(t) = \frac{N^2}{4\Omega^2}\frac{\lambda^2R^2}{\mu_m^2}\cdot\frac{1}{k} = \underbrace{\frac{N^2}{4\Omega_0^2}\frac{R^2\lambda^2}{\mu_m^2}\frac{1}{k}}_{\alpha}\left(\frac{t}{t_0}\right)$$

(10) Using separation of variables,

$$\frac{dV_n}{V_n} = -\frac{dt}{\tau_n(t)} = -\frac{dt}{\alpha\left(\frac{t}{t_0}\right)} \Rightarrow -\frac{t_0}{\alpha}\ln t + \text{const} = \ln V_n$$

$$\Rightarrow V_n = V_n(0)\,t^{-\frac{t_0}{\alpha}}$$

$$\rightarrow \quad V(z,t) = \sum_n V_n(t) Z_n(z) = \sum_n{}' V_n(0) t^{-\frac{t_0}{\alpha}} Z_n(z)$$

Applying ICs:

$$V_0(z) = \sum_n{}' V_n(0) Z_n(z)$$

Projecting onto the Z_n's

$$\int_0^H V_0(z) Z_n(z) = \sum_m{}' V_m(0) \int_0^H Z_n(z) Z_m(z)\,dz$$

$$= V_n(0) \|Z_n(z)\|^2$$

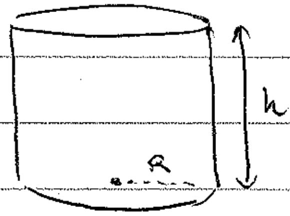
$$\text{so} \quad V_n(0) = \frac{\int_0^H V_0(z) Z_n(z)\,dz}{\|Z_n(z)\|^2}$$

and finally

$$V(z,t) = \sum{}' \frac{\int_0^H V_0(z) Z_n(z)\,dz}{\|Z_n\|^2} t^{-\frac{t_0}{\alpha}} Z_n(z)$$

y

Problem 3



Model:

$$\frac{\partial^2 u}{\partial t^2} = c^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2}\right]$$

(axial symmetry → we can ignore the θ dependence

BCs: for instance take $u=0$ on all boundaries.

Separation of variables: $u(r,z,t) = U(r,z)B(t)$

so

$$\begin{cases} \dfrac{d^2B}{dt^2} = -c^2\lambda^2 B \\ -\omega^2 U = \dfrac{1}{r}\dfrac{d}{dr}\left(r\dfrac{dU}{dr}\right) + \dfrac{d^2U}{dz^2} \end{cases}$$

Then let $U(r,z) = A(r)C(z)$

$$\rightarrow \frac{1}{A}\frac{1}{r}\frac{d}{dr}\left(r\frac{dA}{dr}\right) + \frac{1}{C}\frac{d^2C}{dz^2} = -\lambda^2$$

20 So
$$\begin{cases} \dfrac{1}{r}\dfrac{d}{dr}\left(r\dfrac{dA}{dr}\right) = (-\lambda^2 + k^2)A \\ \dfrac{d^2C}{dz^2} = -k^2 C \end{cases}$$

chose $-k^2$ to satisfy BCs, to get oscillatory functions

The solutions of the C equation, with these BCs, are

20 $C_n(z) = \sin\left(\frac{n\pi z}{L}\right)$ where $k_n = \frac{n\pi}{L}$

The A-equation can be expanded as:

$$\frac{d^2A}{dr^2} + \frac{1}{r}\frac{dA}{dr} + (\lambda^2 - k^2)A = 0$$

$$\Rightarrow r^2\frac{d^2A}{dr^2} + r\frac{dA}{dr} + r^2(\lambda^2 - k^2)A = 0$$

This is a Bessel equation as long as we define $x^2 = r^2(\lambda^2 - k^2)$ then

$$x^2 \frac{d^2A}{dx^2} + x\frac{dA}{dx} + x^2 A = 0$$

→ this is the equation for the $\begin{cases} J_0(x) \\ Y_0(x) \end{cases}$ functions

20 By regularity, we only keep $J_0(x)$ so

$$A(r) = J_0\left(r(\lambda^2 - k^2)^{1/2}\right)$$

Applying BCs on r: $A(R) = 0$ so

20 $R(\omega^2 - k^2)^{1/2} = j_{0m} \in$ zeros of J_0

so

$$\lambda_{nm}^2 - k_n^2 = \frac{j_{0m}^2}{R^2} \text{ so } \lambda_{nm}^2 = k_n^2 + \frac{j_{0m}^2}{R^2}$$

⇒

$$\lambda_{nm} = \sqrt{\frac{n^2\pi^2}{L^2} + \frac{j_{0m}^2}{R^2}}$$

Finally solving the B-equation we get

$$B_{nm}(t) = \begin{cases} \cos(c\lambda_{nm} t) \\ \sin(c\lambda_{nm} t) \end{cases}$$ so the eigenfrequencies

20 are $\omega_{nm} = c\sqrt{\frac{n^2\pi^2}{L^2} + \frac{j_{0m}^2}{R^2}}$