AMS 212A Final 2013

There are 3 problems on this final. Each is worth 35 points. Any substantially completed answer gives you 5 points extra credit, so that answering 2 problems *perfectly* is sufficient for an A.

Calculators are not allowed. You need to justify all your answers. Answers without justifications will be counted as wrong. If you have any hesitation about the question, if you think there may be a problem with it or if you just need clarification, DON'T HESITATE TO ASK!

Problem 1: The Yule Process

Let's consider the following model for a population of bacteria growing in a petri dish. Let $X(t)$ be the number of bacteria present in the dish at time t, and let $P_n(t)$ be the probability that $X(t) = n$. We are going to make the following assumptions:

- Bacteria do not die.
- A single bacteria gives birth to another bacteria (by cellular division) with probability $\lambda h + o(h)$ during the time interval $(t, t+h)$. The probability that the same bacteria (or its offspring) gives birth to yet another one during the same interval of time is negligible.
- Bacteria give birth independently of one another.
- At time $t = 0$, there are n_0 bacteria in the dish. We know that for sure!

(a) Show that the probabilities $P_n(t)$ satisfy the coupled system of ODEs:

$$
\begin{aligned}
&P'_n(t) = 0 \text{ for } n < n_0 \\
&P'_{n_0}(t) = -\lambda n_0 P_{n_0}(t) \\
&P'_n(t) = -\lambda n P_n(t) + \lambda(n-1)P_{n-1}(t) \text{ for } n > n_0
\end{aligned}
$$

(b) Show that the probability generating function $G(t, y) = \sum_n P_n(t) y^n$ satisfies the initial value problem

$$
\begin{aligned}
&\frac{\partial G}{\partial t} = -y\lambda(1-y)\frac{\partial G}{\partial y} \\
&G(0, y) = y^{n_0}
\end{aligned}
$$

(c) Solve this problem and show that

$$
G(y, t) = \left[\frac{ye^{-\lambda t}}{1 - y(1 - e^{-\lambda t})}\right]^{n_0}
$$

(d) Deduce that the expectation value of the population size is $n_0 e^{\lambda t}$, and comment on whether this is the result you expected.

Problem 2: Spreading of dye

A scientist gradually releases a powerful dye at the center of a Petri dish of radius R, and observes how the dye concentration profile $C(r, \theta, t)$ spreads. The process is well-described by the following model:

$$\begin{aligned}
\frac{\partial C}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 C}{\partial \theta^2} &+ S(t)\delta(r) \\
C(r, \theta, 0) &= 0 \\
\left.\frac{\partial C}{\partial r}\right|_{(R,\theta,t)} &= 0 \\
S(t) &= S_0 e^{-kt}
\end{aligned} \tag{1}$$

The $\delta-$function considered here is a 2-dimensional $\delta-$function, which satisfies:

$$2\pi \int_0^\infty r\delta(r) f(r) dr = f(0)$$

for any function f.

(a) By symmetry considerations, simplify the problem. Explain your argument carefully.

(b) Verify, in any way you want, that the total amount of dye in the system is conserved, that is,

$$\int_0^R 2\pi r C(r, t) dr = \int_0^t S_0 e^{-kt'} dt'$$

Explain what this equation means, in words.

(c) Show that the solution can be written as

$$C(r, \theta, t) = \sum_n a_n(t) J_0\left(\frac{j_{1n}}{R} r\right)$$

where the j_{1n} coefficients are the zeros of the J_1 Bessel function. What equation and initial condition do the $a_n(t)$ functions satisfy?

(d) Solve this equation, and write the complete solution. Evaluate all remaining integrals using the formulas below.

For questions (b) and (c), you will probably need to use the fact that :

- The equation $x^2 f'' + x f' + (x^2 - n^2) f = 0$ is a Bessel equation of order n, and has solutions $J_n(x)$ (which is regular) and $Y_n(x)$ (which is singular).
- $$J_0(0) = 1$$
- $$\frac{dJ_0}{dx} = -J_1(x)$$
- $$\int_0^R x J_0^2\left(\frac{j_{1n}}{R} x\right) dx = \frac{1}{2} R^2 J_0^2(j_{1n})$$

Problem 3: d'Alembert's solution

Consider the following problem:

$$\begin{aligned} u_{tt} = c^2 u_{xx} \text{ for } x \in (-\infty, +\infty) \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{aligned}$$

(a) Is it hyperbolic, parabolic or elliptic?

(b) Using canonical forms, recover d'Alembert's solution:

$$u(x,t) = \frac{1}{2}\left(f(x-ct) + f(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(x')dx'$$

Do not just plug in the solution to see if it works – derive it from scratch, justifying all of your steps.