Final, AMS212A 2009

Each question is worth 50 points. You are allowed your own notes, homeworks, and textbook. Justify every step of your calculations.

Problem 1: Time-dependent boundary conditions

This problems guides you through the method used towards solving linear PDEs with time-dependent boundary conditions, using an example based on the diffusion equation.

Consider the following problem:

$$\begin{aligned} u_t &= k u_{xx} \\ u(0,t) &= 0 \\ u(L,t) &= f(t) \\ u(x,0) &= u_0(x) \end{aligned} \tag{1}$$

(1) Construct a function $v(x,t) = u(x,t) - h(x)f(t)$. What conditions does the function $h(x)$ have to satisfy to ensure that

$$\begin{aligned} v(0,t) &= 0 \\ v(L,t) &= 0 \end{aligned} \tag{2}$$

(2) Choose the simplest possible function $h(x)$ which has these properties, and show that the PDE and associated initial conditions satisfied by $v(x,t)$ are

$$\begin{aligned} v_t &= k v_{xx} + F(x,t) \\ v(0,t) &= 0 \\ v(L,t) &= 0 \\ v(x,0) &= v_0(x) \end{aligned} \tag{3}$$

where you must express $F(x,t)$ and $v_0(x)$ in terms of known quantities.

(3) Find a formal solution of this forced problem for $v(x,t)$ in terms of the functions $F(x,t)$ and $v_0(x)$. Justify every step of your answer. If you are using a Greens' function, you must justify every step of its construction.

(4) Apply your formal solution to $u_0(x) = 0$ and $f(t) = 1 - e^{-t}$, and deduce $u(x,t)$.

(5) What is the limit of the solution $u(x,t)$ as $t \to \infty$? Interpret this result physically, by describing a physical experiment which could be modeled using the above PDE, initial and boundary conditions.

Problem 2: Radial stellar oscillations

Many stars are known to *pulsate*, i.e. to undergo purely periodic radial pressure oscillations. These stellar pulsations are modeled using the wave equation in spherical coordinates,

$$\frac{\partial^2 P}{\partial t^2} = c^2(r)\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial P}{\partial r}\right)\right] \tag{4}$$

These are regular pulsations (i.e. the star doesn't blow up) so that the pressure always remains bounded. The pressure at the surface of the star (at $r = R$) is assumed to be zero, and $\mathrm{d}P/\mathrm{d}r = 0$ at $r = 0$.

(1) Using separation of variables, find the spatial eigenvalue Sturm-Liouville problem associated to this PDE. What are $p(r)$, $q(r)$, and the weight function $w(r)$ (in the standard notation associated with Sturm-Liouville problems). Is this a regular or a singular problem?

(2) Justify *mathematically* the sign of the eigenvalues using properties of the Rayleigh Quotient.

(3) Assuming that the sound-speed inside the star $c(r)$ is constant and equal to c_0, provide an *estimate* for the fundamental eigenvalue, and therefore deduce an estimate for the *period* of the fundamental oscillation of a star with radius 10^{11}cm, and $c_0 \simeq 10^7$cm/s (recall that period = 2π/frequency).

(4) Again, assuming that the sound speed inside the star is constant and equal to c_0, show that the solutions to this Sturm-Liouville problem are Spherical Bessel Functions. You will need to use the fact that the Spherical Bessel equation is

$$x^2 f'' + 2xf' + (x^2 - n(n+1))f = 0 \tag{5}$$

and has two types of solutions: $j_n(x)$ which are regular at $x = 0$ and $y_n(x)$ which are singular at $x = 0$.

(5) Show that $j_0(x) = \sin(x)/x$ (you will need to check that it is a solution of the relevant equation and that it is regular at $x = 0$).

(6) Deduce the exact value of the fundamental eigenvalue, and therefore of the fundamental period of the star. How far off was your earlier estimate?

(7) Let us now go back to the general case of c being a function of r. A rough approximation to $c(r)$ is given by

$$c(r) = c_s + c_m\left(1 - \frac{r}{R}\right) \tag{6}$$

where $c_s = 10^5$cm/s and $c_m = 2.5 \times 10^7$ cm/s. Write an approximate estimate for the eigenfrequencies of oscillation for large n.