Final

Instructions:

- This is an open book exam. Any notes from this class are allowed.
- All problems are worth the same number of points. Attempt as many as you can, bearing in mind that fully and correctly completed problems will be awarded extra credit.
- Read all the problems first before beginning to answer any of them. Start with the one you feel most comfortable with, and only move on to the next problem when you are certain you have completed it perfectly.
- Calculators are allowed but shouldn't be necessary.
- MAKE SURE YOUR NAME IS ON THE FIRST PAGE OF YOUR ANSWERS, AND YOUR INITIALS ON ALL OTHER PAGES
- The instructor reserves the right not to grade any illegible answers.
- Note: solving 3 problems completely and correctly will garantee an A^+.
- RELAX, AND GOOD LUCK!

Problem 1: This problem is aimed at studying the response of the Golden Gate bridge to wind forcing. Let us model the bridge as a one-dimensional "elastic" object of length L, attached to the land at two fixed points A and B, and call x the distance along the bridge from its southernmost point. Only lateral motions are allowed (y-direction). The equation of motion of any point on the bridge is therefore given by the equation

$$u_{tt} - c^2 u_{xx} = W(x,t)$$
$$u(0,t) = u(L,t) = 0$$

where $W(x,t)$ is a function describing the wind forcing term. We assume that the wind blows into the Bay with a velocity that varies both along the bridge and with time:

$$W(x,t) = x(L-x)\sin^2(\omega t) \tag{1}$$

Finally, we assume that at time $t = 0$ the bridge is at rest and in its equilibrium position:

$$u(x,0) = 0$$
$$u_t(x,0) = 0$$

- Calculate the spatial eigenmodes of the homogeneous wave equation satisfying the same boundary conditions, and deduce that the solution of the forced problem can be written as

$$u(x,t) = \sum_n^{\infty} f_n(t)\sin(k_n x) \tag{2}$$

  where you must determine k_n explicitly,

- Show that the functions $f_n(t)$ satisfy the ODEs

$$f_n'' + c^2 k_n^2 f_n = \alpha_n \sin^2(\omega t) \tag{3}$$

  where

$$\alpha_n = \frac{4L^2}{n^3\pi^3}(1-\cos(n\pi)) \tag{4}$$

- How do the initial conditions for $u(x,t)$ relate to $f_n(0)$ and $f_n'(0)$?

- Deduce that

$$f_n(t) = a_n(t) + b_n + d_n\cos(2\omega t) \tag{5}$$

  where you must determine the function $a_n(t)$, and the constants b_n and d_n. (Hint: use the identity $\sin^2 x = (1-\cos(2x))/2$).

- Write the complete solution $u(x,t)$

Problem 2: This problem is aimed at finding the equilibrium temperature profile within a spherical cavity. We consider a region bounded by two concentric spheres of radius a and b respectively (with $a < b$). We use a spherical coordinate system (r, θ, ϕ). The temperature on the inner sphere is held at a constant value T_0

$$T(a, \theta, \phi) = T_0 \tag{6}$$

while the temperature on the outer sphere is

$$T(b, \theta, \phi) = T_1 + T_2 \cos(\theta) \tag{7}$$

The steady-state temperature profile is obtained by solving

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \phi^2} = 0 \tag{8}$$

subject to the boundary conditions required.

- Verify that the functions $v_0(\theta) = 1$ and $v_1(\theta) = \cos\theta$ are eigen-solutions of the equation

  $$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta \frac{\mathrm{d}v_i}{\mathrm{d}\theta}\right) = -\lambda_i v_i \tag{9}$$

  and find the corresponding eigenvalues λ_0 and λ_1.

- Give a solid argument why the solution to the problem can be written as

  $$T(r, \theta, \phi) = f_0(r) + f_1(r)\cos\theta \tag{10}$$

  where $f_0(r)$ and $f_1(r)$ satisfy (for $i = 0$ or $i = 1$)

  $$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}f_i}{\mathrm{d}r}\right) - \frac{\lambda_i}{r^2} f_i = 0 \tag{11}$$

- Solve the equations for $f_0(r)$ and $f_1(r)$ and identify the arbitrary constants that will need to be determined from the boundary conditions. (Hint: seek solutions in r^α).

- By applying the boundary conditions, find $T(r, \theta, \phi)$.

Problem 3: Solve the problem

$$u_{tt} - 4u_{xx} = e^x + \sin t, \qquad -\infty < x < \infty, t > 0$$
$$u(x,0) = 0, \qquad -\infty < x < \infty$$
$$u_t(x,0) = \frac{1}{1+x^2}, \qquad -\infty < x < \infty$$

Problem 4: To solve the problem

$$u_{xx} + (1+y^2)^2 u_{yy} + 2y(1+y^2)u_y = 0$$
$$u(x,0) = x$$
$$u_y(x,0) = -2x$$

- Show that this equation is equivalent to Laplace's equation in a different coordinate system (ξ, η).
- Find the general solution. Hint: the boundary conditions suggest that you try a bilinear form in (ξ, η) as a solution of the Laplace equation.