Final 2013

Problem 1

8 (a) • If bacteria can't die, then $n \geq n_0$ always

so $P_n'(t) = 0$ if $n < n_0$ [2]

• $P_{n_0}'(t)$?

$P_{n_0}(t+h) = P_{n_0}(t) \cdot$ probability that nothing happens

$= P_{n_0}(t)(1 - \lambda n_0 h)$

↑ Probability that one bacteria divides.

So $\dfrac{P_{n_0}(t+h) - P_{n_0}(t)}{h} = -\lambda n_0 P_{n_0}$

$\rightarrow P_{n_0}'(t) = -\lambda n_0 P_{n_0}$ [3]

• $P_n'(t)$?

$P_n(t+h) = P_{n-1}(t) \cdot$ Probability that one divided

$+ P_n(t) \cdot$ Probability that nothing happened

$= P_{n-1}(t) \cdot \lambda(n-1)h$

$+ P_n(t) \cdot (1 - \lambda n h)$

$\Rightarrow P_n'(t) = \dfrac{P_n(t+h) - P_n(t)}{h}$

$= \dfrac{1}{h}\left\{ P_{n-1}(t)\lambda(n-1)h + P_n(t)(1-\lambda n h) - P_n(t) \right\}$

$P_n'(t) = P_{n-1}\lambda(n-1) - P_n \lambda n$ [3].

8 (b) let $G = \sum_{n=0}^{\infty} P_n(t) g^n = \sum_{n=n_0}^{\infty} P_n(t) g^n$ 2/

$$\frac{\partial G}{\partial t} = \sum_{n=n_0}^{\infty} P_n'(t) g^n \quad [1] \qquad \frac{\partial G}{\partial g} = \sum_{n=n_0}^{\infty} n P_n(t) g^{n-1} \quad [1]$$

$$= \sum_{n_0+1}^{\infty} P_n'(t) g^n + \left(-\lambda n_0 P_{n_0}(t) g^{n_0}\right)$$

$$= \sum_{n_0+1}^{\infty} g^n \left(P_{n-1}\lambda(n-1) - \lambda n P_n\right) - \lambda n_0 P_{n_0}(t) g^{n_0}$$

$$= -\lambda g \sum_{n_0}^{\infty} n P_n g^{n-1} + \sum_{n_0+1}^{\infty} g^n \lambda(n-1) P_{n-1}$$

$m = n-1$

$$= -\lambda g \frac{\partial G}{\partial g} + \sum_{m=n_0}^{\infty} g^{m+1} \lambda m P_m$$

$$= -\lambda g \frac{\partial G}{\partial g} + \lambda g^2 \frac{\partial G}{\partial g} = \lambda g (g-1) \frac{\partial G}{\partial g}$$

$$= -\lambda g (1-g) \frac{\partial G}{\partial g} \qquad [5]$$

$$G(0,g) = \sum_{n=n_0}^{\infty} P_n(0) g^n = P_{n_0}(t) g^{n_0} = g^{n_0} \quad [1] \qquad \text{since } P_n(0) = 0 \ \forall n > n_0, \ n < n_0$$

10 (c) let's solve $\frac{\partial G}{\partial t} + \lambda y(1-y)\frac{\partial G}{\partial y} = 0$

$$\begin{cases} \frac{dt}{d\tau} = 1 & \rightarrow t = \tau \quad [1] \\ \frac{dy}{d\tau} = \lambda y(1-y) & \rightarrow y = ? \\ \frac{\partial G}{\partial \tau} = 0 & \rightarrow G = G(0,y) = y^{n_0} \quad [1] \end{cases} \qquad \begin{cases} t_0(s) = 0 \\ y_0(s) = s \\ G_0(s) = s^{n_0} \end{cases} \quad [2]$$

$$\frac{dy}{y(1-y)} = \lambda d\tau. \qquad \text{By partial fractions}$$

$$\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y} = \frac{1}{y} + \frac{1}{1-y}$$

so $$\int \frac{dy}{1-y} = \int \frac{dy}{y} + \int \frac{dy}{1-y}$$

$$= \ln|y| - \ln|1-y|$$

$$= \ln\left|\frac{y}{1-y}\right|$$

$$\Rightarrow \quad \ln\left|\frac{y}{1-y}\right| = \lambda\tau + \text{const} \qquad [3].$$

at $\tau = 0$, $y = s$ so $\text{const} = \ln\left|\frac{s}{1-s}\right|$

so finally, $\left|\frac{y}{1-y} \cdot \frac{1-s}{s}\right| = e^{\lambda\tau}$

To solve for s, note:

$$(1-s)\frac{y}{1-y} = se^{\lambda\tau}$$

$$s\left[e^{\lambda\tau} + \frac{y}{1-y}\right] = \frac{y}{1-y}$$

$$s = \frac{\frac{y}{1-y}}{e^{\lambda\tau} + \frac{y}{1-y}} = \frac{y}{(1-y)e^{\lambda\tau} + y} = \frac{ye^{-\lambda\tau}}{1-y+ye^{-\lambda\tau}}$$

$$= \frac{ye^{-\lambda\tau}}{1-y(1-e^{-\lambda\tau})} \qquad [2]$$

Since $G(s) = s^{n_0}$:

$$G(t,y) = \left[\frac{ye^{-\lambda t}}{1-y(1-e^{-\lambda t})}\right]^{n_0}. \qquad ✓ \qquad [1].$$

6(d) The expectation value is

$$\sum_{n=n_0}^{\infty} P_n(t).n.$$

$$= \left.\frac{\partial G}{\partial y}\right|_{y=1} \qquad [2]$$

$$\ln G = n_0 \left\{ \ln(ye^{-\lambda t}) - \ln(1 - y(1-e^{-\lambda t})) \right\}$$

$$= n_0 \left\{ \ln y - \lambda t - \ln(1 - y(1-e^{-\lambda t})) \right\}$$

$$\frac{1}{G}\frac{\partial G}{\partial y} = \frac{n_0}{y} + \frac{n_0(1-e^{-\lambda t})}{1-y(1-e^{-\lambda t})}$$

$$\left.\frac{\partial G}{\partial y}\right|_{y=1} = G(y=1) \cdot \left(n_0 + n_0\left(\frac{1-e^{-\lambda t}}{1-(1-e^{-\lambda t})}\right)\right)$$

$$= \left[\frac{e^{-\lambda t}}{1-(1-e^{-\lambda t})}\right]^{n_0} \cdot \left(n_0 + n_0 \frac{1-e^{-t\lambda}}{e^{-\lambda t}} \right)$$

$$= \frac{n_0 e^{-\lambda t} + n_0(1-e^{-\lambda t})}{e^{-\lambda t}} = n_0 e^{\lambda t}. \qquad [4].$$

3 This is as expected for a dividing bacterial population, as long as the reproduction probability remains constant & no deaths occur.

Problem 2

(a) The problem is independent of θ
(ICs, BCs and forcing all are) so

$$\frac{\partial c}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial c}{\partial r}\right) + S(t)\delta(r) \qquad [3]$$

(b) See below

(c). Consider the homogeneous problem:

$$\frac{\partial c}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial c}{\partial r}\right)$$

Let $C = A(t)B(r)$

$$\rightarrow \begin{cases} \frac{dA}{dt} = -\lambda A \\ \frac{1}{r}\frac{d}{dr}\left(r\frac{dB}{dr}\right) = -\lambda B \end{cases}$$

$$\Rightarrow \frac{d}{dr}\left(r\frac{dB}{dr}\right) = -\lambda r B$$ $\rightarrow$ a S.L. problem with weight function $w(r) = r$.

$$\Rightarrow r\frac{d^2B}{dr^2} + \frac{dB}{dr} + \lambda r B = 0$$

[6] $$\Rightarrow x^2\frac{d^2B}{dx^2} + x\frac{dB}{dx} + x^2 B = 0 \quad \text{if } x = \sqrt{\lambda}\, r$$

This is a Bessel Equation for $n=0$: so there are two solutions: $\left(\begin{matrix} J_0(\sqrt{\lambda}\, r) \\ Y_0(\sqrt{\lambda}\, r)\end{matrix}\right)$.

To satisfy the regularity condition, we drop Y_0, and to satisfy the BCs, we need $J_0'(\sqrt{\lambda}R) = 0$

Since $J_0'(x) = -J_1(x)$ this implies $\sqrt{\lambda}R = j_{1n}$ $\Rightarrow \lambda = \left(\frac{j_{1n}}{R}\right)^2$

[4]

j_{1n} = zeros of $J_1(x)$

Now consider that

[2] $$C(r,t) = \sum_n{}' A_n(t)\, J_0\left(\frac{\gamma_{1n}}{R} r\right)$$

Plugging this back into the main equation we get

$$\sum_n{}' \frac{dA_n}{dt} J_0\left(\frac{\gamma_{0n}}{R} r\right) = \sum_n{}' A_n(t) \cdot \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\left(J_0\left(\frac{\gamma_{1n}}{R} r\right)\right)\right)$$
$$+ S(t)\delta(r)$$

(*) $$= \sum_n{}' A_n(t) \cdot \left(-\frac{\gamma_{1n}^2}{R^2}\right) J_0\left(\frac{\gamma_{1n}}{R} r\right) + S(t)\delta(r)$$

Given that $\int_0^R r\, J_0\left(\frac{\gamma_{1n}}{R} r\right) J_0\left(\frac{\gamma_{1m}}{R} r\right) dr = 0$ if $m \neq n$ (Using properties of S.L problem and the fact that weight function is $N(r) = r$

and $\int_0^R r\, J_0^2\left(\frac{\gamma_{1n}}{R}\right) dr = \frac{1}{2} R^2 J_0^2(\gamma_{1n})$

then, integrating the governing eq(*) we get:

$$\frac{dA_n}{dt} = -\frac{\gamma_{1n}^2}{R^2} A_n + S(t) \frac{\int_0^R r\,\delta(r) J_0\left(\frac{\gamma_{1n}}{R} r\right) dr}{}$$

$$= -\frac{\gamma_{1n}^2}{R^2} A_n + S(t) \frac{\frac{1}{2\pi} J_0(0)}{\frac{1}{2} R^2 J_0^2(\gamma_{1n})} \qquad J_0(0) = 1$$

[6] $$= -\frac{\gamma_{1n}^2}{R^2} A_n + \frac{1}{\pi R^2} \frac{S(t)}{J_0^2(\gamma_{1n})}$$

This can be solve w. an integrating factor method.

Let $\mu(t) = e^{(\gamma_{1n}^2/R^2)t}$

then $$\frac{d}{dt}(A_n \mu) = \frac{\mu(t) S(t)}{\pi R^2 J_0^2(\gamma_{1n})} = \frac{S_0 e^{(\gamma_{1n}^2/R^2 - k)t}}{\pi R^2 J_0^2(\gamma_{1n})}$$

$$\Rightarrow \quad A_n(t)\mu(t) - A_n(0)\mu(0) = \frac{1}{\pi R^2 J_0^2(j_{in})}\left[\frac{S_0 e^{\left(\frac{j_{in}^2}{R^2}-k\right)t}}{\frac{j_{in}^2}{R^2}-k}\right]_0^t$$

$$\Rightarrow \quad A_n(t) = A_n(0)\cdot e^{-\left(\frac{j_{in}^2}{R^2}\right)t} + \frac{S_0 e^{-\frac{j_{in}^2}{R^2}t}}{\pi R^2 J_0^2(j_{in})}\left\{\frac{e^{\frac{j_{in}^2}{R^2}t-kt}-1}{\frac{j_{in}^2}{R^2}-k}\right\}$$

[6] $$A_n(t) = A_n(0)e^{-\left(\frac{j_{in}^2}{R^2}\right)t} + \frac{S_0\left(e^{-kt} - e^{-\frac{j_{in}^2}{R^2}t}\right)}{\pi R^2 J_0^2(j_{in})\left(\frac{j_{in}^2}{R^2}-k\right)}$$

So finally:

$$C(r,t) = \sum_n \left[A_n(0)e^{-\left(\frac{j_{in}}{R}\right)^2 t} + S_0\frac{e^{-kt} - e^{-\frac{j_{in}^2}{R^2}t}}{\pi R^2 J_0^2(j_{in})\left(\frac{j_{in}^2}{R^2}-k\right)}\right] J_0\left(\frac{j_{in}}{R}r\right)$$

[3] At $t=0$, $C(r,0)=0$ so $A_n(0)=0$ too.

$\Rightarrow$ This implies

$$C(r,t) = \frac{S_0}{\pi R^2}\sum_n \frac{e^{-kt} - e^{-\left(\frac{j_{in}}{R}\right)^2 t}}{J_0^2(j_{in})\left(\frac{j_{in}^2}{R^2}-k\right)} J_0\left(\frac{j_{in}}{R}r\right)$$

(b) Consider

[5] $$2\pi r\frac{\partial C}{\partial t} = 2\pi\frac{\partial}{\partial r}\left(r\frac{\partial C}{\partial r}\right) + 2\pi r\,\delta(r)S(t)$$

Then integrate between 0 and R:

$$\int 2\pi r\frac{\partial C}{\partial t}\,dr = 2\pi\cancel{\left[r\frac{\partial C}{\partial r}\right]_0^R}^{0 \text{ by BCs}} + S(t)$$ by definition of the δ - function

$$\Rightarrow \frac{\partial}{\partial t}\int_0^R 2\pi r C\,dr = S(t)$$

$$\Rightarrow \quad 2\pi r C(r,t) - \cancel{2\pi r C(r,0)}^{0 \text{ by ICs}} = \int_0^t S(t')\,dt'$$

$$2\pi r\,C(r,t) = \int_0^t S_0 e^{-kt'}\,dt'$$ as required.

Problem 3

$$u_{tt} - c^2 u_{xx} = 0$$
$$u(x,0) = f(x)$$
$$u_t(x,0) = g(x)$$

- $\delta = 0 - 1\cdot(-c^2) = c^2 > 0$ so hyperbolic [2]

- let $\frac{dx}{dt} = \frac{0 \pm \sqrt{\delta}}{1} = \pm\sqrt{c^2} = \pm c$ [4]

$x(t) = \pm ct + \text{const}.$

$\Rightarrow$ on ξ- characteristics let $x + ct = \xi$ [4]
on η- ------- let $x - ct = \eta$

- Then: $\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$

$\frac{\partial}{\partial t} = \frac{\partial}{\partial \xi}\cdot\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial \eta}\cdot\frac{\partial \eta}{\partial t} = c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta}$

$$u_{xx} = \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)^2 u = \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} \quad [2]$$

$$u_{tt} = \left(c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta}\right)^2 u = c^2\frac{\partial^2 u}{\partial \xi^2} - 2c^2\frac{\partial^2 u}{\partial \xi \partial \eta} + c^2\frac{\partial^2 u}{\partial \eta^2} \quad [2]$$

$$u_{tt} - c^2 u_{xx} = c^2\cancel{\frac{\partial^2 u}{\partial \xi^2}} - 2c^2\frac{\partial^2 u}{\partial \xi \partial \eta} + c^2\cancel{\frac{\partial^2 u}{\partial \eta^2}} - c^2\left(\cancel{\frac{\partial^2 u}{\partial \xi^2}} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \cancel{\frac{\partial^2 u}{\partial \eta^2}}\right)$$

$$= -4c^2\frac{\partial^2 u}{\partial \xi \partial \eta} = 0 \quad [2]$$

$\Rightarrow u_{\xi\eta} = 0 \Rightarrow u(\xi,\eta) = F(\xi) + G(\eta)$
$u(x,t) = F(x+ct) + G(x-ct).$

[4]

Applying ICs:

at $t=0$ $\quad u(x,0) = f(x) = F(x) + G(x)$ [2]

$u_t(x,0) = g(x) = cF'(x) - cG'(x)$ [2]

so $\quad F' - G'(x) = \dfrac{g(x)}{c}$

$$\Rightarrow \quad F(x) - G(x) - (F(0) - G(0)) = \int_0^x \frac{g(x')}{c}\,dx' \qquad [4]$$

Combining this with $f(x) = F(x) + G(x)$

$$\Rightarrow \quad 2F(x) = F(0) + G(0) + \int_0^x \frac{g(x')}{c}\,dx' + f(x) \qquad [2]$$

$$\Rightarrow \quad 2G(x) = -F(0) - G(0) - \int_0^x \frac{g(x')}{c}\,dx' + f(x) \qquad [2]$$

$$\Rightarrow \quad u(x,t) = \frac{1}{2}\cancel{(F(0) + G(0))} + \frac{1}{2}\int_0^{x+ct} \frac{g(x')}{c}\,dx' + \frac{1}{2}f(x+ct)$$

$$- \frac{1}{2}\cancel{(F(0) + G(0))} - \frac{1}{2}\int_0^{x-ct} \frac{g(x')}{c}\,dx' + \frac{1}{2}f(x-ct)$$

$$= \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2}\int_{x-ct}^{x+ct} \frac{g(x')}{c}\,dx'.$$

[3]