Take Home Midterm 2012

If you think there is a typo, please email/call me asap (459-1055). To be handed in on Monday, Feb. 20, 10am. Scan it/email it if you cannot drop it in my office. No delays accepted.

You need to justify all your answers. Answers without justifications will be counted as wrong. You **may not** ask for help from any other living being aside from the instructor. Despite anecdotal evidence to the contrary, graduate students, postdocs and faculty always count as living beings. If you need help from the instructor, make sure to ask **before** Friday 6pm. Help will be limited to clarifications on the questions if needed.

Problem 1:

A cubic container of size $1 \times 1 \times 1$ is filled, very, very carefully (so that no motion mix the fluid), half-way up with cold water (at temperature $T_c = 10°\text{C}$) and the rest of the way up with hot water (at temperature $T_h = 50°\text{C}$). At $t = 0$, the container is placed in bath of very cold water ($T_b = 0°\text{C}$), to make sure the sides are held at $0°\text{C}$ from $t = 0$ onwards.

(a) Describe the mathematical system of equations and associated conditions which describes this experiment. You may assume that the thermal diffusivity is 1.

(b) Explain why the solution will depend on all three spatial variables.

(c) Solve this problem to find the evolution of the temperature profile inside the box. What is the global diffusion timescale for this problem?

Problem 2:

A cylindrical container of volume $V = 1$ is subject to exactly the same treatment as the cube of Problem 1. Let R be its radius, and h its height.

(a) What is the relationship between h and R?

(b) What equation and associated conditions does the system satisfy?

(c) Show that the respective diffusion timescales of individual spatial eigenmodes of the cylinder are

$$\tau_{lmn} = \left(\frac{l^2\pi^2}{h^2} + \left(\frac{z_{nm}}{R} \right)^2 \right)^{-1}$$

where z_{nm} is the m-th zero of the n-th Bessel J function.

(d) Find the longest diffusion timescale among all possible modes.

(e) Determine whether the overall diffusion timescale of the cube is (i) always larger (ii) always smaller or (iii) sometimes equal to that of the cylinder. If the answer is (iii) give the value(s) of the aspect ratio for which equality holds. Hint: you will probably need to use a graphing tool to answer this question. Analytical solutions are not required. Conclude on the optimal shape of water containers to preserve heat as long as possible.

Problem 3:

This problem helps you find the eigenfrequencies of radial oscillations of a Saturn-like planet. It is related to a problem on stellar oscillations in a recent year Final problem set.

Let's assume the planet has radius $R = 60,000$km $= 6 \times 10^9$ cm. There is a solid inner core that spans about 20% of the planet's radius, from $r = 0$ to $r = 0.2R$. The region above is gaseous, and is composed mostly of Hydrogen and Helium. Radial pressure waves propagate in the gaseous part of the planet, following the equation (valid for $0.2R < r < R$)

$$\frac{\partial^2 p}{\partial t^2} = \frac{c_0^2}{r^2} \frac{\partial}{\partial r} \left[r^2 \frac{\partial p}{\partial r} \right]$$

(in this problem we will neglect any θ and ϕ dependence). The pressure perturbations $p(r, t)$ are assumed to be zero at the boundary with the inner core, and at the outer boundary of the planet. We will assume that the sound speed in the gas is more or less constant, with value $c_0 = 2 \times 10^6$ cm/s.

(a) Using separation of variables, identify the equation satisfied by the radial eigenmodes. Is this a Sturm Liouville problem? Explain. Is this a regular or a singular problem?

(b) Find the general solution of the radial problem. Hint: the stellar oscillation problem is very similar to this one, and will help you find the correct set of basic solutions to the equation.

(c) By fitting the boundary conditions (Beware! this is where things get tricky), show that the eigenfrequencies of the planet are related to the zeros of a particular function $f(x)$. What is this function? By plotting $f(x)$ using Mathematica or Maple, find numerical estimates to the first 5 zeros.

(d) Deduce the period of oscillation of the fundamental and first radial harmonic modes (in hours)

Problem 4:

Problem 8.5.6 of the textbook. Note: partial answers are found in the book, so you know what to aim for. Justifications for all the steps are required.