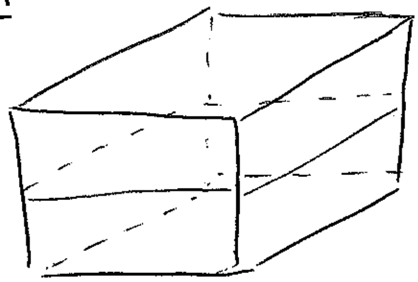
MIDTERM 2012 25 pts/question; +5 extra credit for "perfect" answers. 1.

[25] Problem 1



(a) Mathematical model. [3]

- $\frac{\partial T}{\partial t} = k\nabla^2 T = \nabla^2 T$ if $k=1$

  $= T_{xx} + T_{yy} + T_{zz}$ in cartesian system

- $T = 0$ on all sides of cube

- $T(x,y,z,0) = \begin{cases} 10 & \text{if } z < \frac{1}{2} \\ 50 & \text{if } z > \frac{1}{2} \end{cases}$

[2] (b) • At time $t=0$, say $T(x,y,z,0) = 10$ for $z < \frac{1}{2}$ and $x \in (0,1)$

but $T = 0$ at $x = 0$ and $= 1$

$\rightarrow$ the system is not independent of x.

- similarly for y
- System clearly depends on z.

(c) let $T(x,y,z,t) = A(x,y,z)\,B(t)$

$$\rightarrow \begin{cases} B_t = -\lambda B \\ A_{xx} + A_{yy} + A_{zz} = -\lambda A \end{cases}$$ we expect $\lambda > 0$ to avoid exponentially growing solutions

let $A_{xx} + A_{yy} = -A_{zz} - \lambda A$

if $A(x,y,z) = \alpha(x)\,\beta(y)\,\gamma(z)$ then

$$\gamma\left(\beta\alpha_{xx} + \alpha\beta_{yy}\right) = -\alpha\beta\gamma_{zz} - \alpha\beta\gamma\lambda$$

$$\Rightarrow \frac{\alpha_{xx}}{\alpha} + \frac{\beta_{yy}}{\beta} = -\frac{\gamma_{zz}}{\gamma} - \lambda = \text{constant } K$$

$$\Rightarrow \gamma_{zz} = (-\lambda - K)\gamma$$

Then $\frac{\alpha_{xx}}{\alpha} + \frac{\beta_{yy}}{\beta} = K$

$$\Rightarrow \quad \frac{\alpha_{xx}}{\alpha} = K - \frac{\beta_{yy}}{\beta} = -C^2$$

expect a negative constant to fit BCs. (homogeneous, T=0 in x)

$$\Rightarrow \quad \begin{cases} \alpha_{xx} = -C^2\alpha \\ \beta_{yy} = (K+C^2)\beta \\ \gamma_{zz} = -(\lambda+K)\gamma \end{cases} \quad \rightarrow [5]$$

- To fit the x- boundary conditions we expect

$$\alpha_n(x) = \sin(n\pi x) \quad \text{and} \quad C_n = n\pi \quad [2]$$

- To fit the y- boundary conditions, we need $K + C_n^2 < 0$ too, in which case we pick up another sin function for β:

$$\beta_m(y) = \sin(m\pi y)$$

$$\text{and} \quad K_{nm} + C_n^2 = -m^2\pi^2$$

$$\text{so} \quad K_{nm} = -(n^2+m^2)\pi^2 \quad [2]$$

- To fit the z- boundary conditions we need $-\lambda - K$ negative, in which case we pick up a third "sin" function:

$$\gamma_\ell(z) = \sin(\ell\pi z)$$

$$\text{and} \quad -\lambda_{\ell nm} - K_{nm} = -\ell^2\pi^2$$

$$\text{so} \quad \lambda_{\ell nm} = (\ell^2+n^2+m^2)\pi^2 \quad [2]$$

$\Rightarrow$ finally we get

$$T(x,y,z,t) = \sum_{\ell nm} d'_{\ell nm} \sin(n\pi x)\sin(m\pi y)\sin(\ell\pi z)\, e^{-\lambda_{\ell nm} t} \quad [2]$$

How to fit the initial conditions?

We have

$$T(x,y,z,0) = \sum_{\ell nm}' \alpha_{nm\ell} \sin(n\pi x)\sin(m\pi y)\sin(\ell\pi z) = \begin{cases} 10 & \text{if } z<\frac{1}{2} \\ 50 & \text{if } z>\frac{1}{2} \end{cases}$$

$$\rightarrow \quad \alpha_{nm\ell} = \frac{\int_0^1\int_0^1\int_0^1 \sin(n\pi x)\sin(m\pi y)\sin(\ell\pi z)\cdot T_0(x,y,z)\,dx\,dy\,dz}{\int_0^1\int_0^1\int_0^1 \sin^2(n\pi x)\sin^2(m\pi y)\sin^2(\ell\pi z)\,dx\,dy\,dz}$$

[2].

The integrals separate:

$$\int_0^1 \sin(n\pi x)dx = \frac{-1}{n\pi}\left[\cos(n\pi x)\right]_0^1$$

$$= -\frac{1}{n\pi}\left(\cos(n\pi)-1\right)$$

$$= -\frac{1}{n\pi}\left((-1)^n-1\right) = \begin{cases} \frac{2}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$\int_0^1 \sin^2(n\pi x)dx = \frac{1}{2}$$; similarly for y.

$$10\int_0^{\frac{1}{2}} \sin(\ell\pi z)dz + \int_{\frac{1}{2}}^1 50\sin(\ell\pi z)dz = \frac{-10}{\ell\pi}\left[\cos\left(\frac{\ell\pi}{2}\right)-1\right] - \frac{50}{\ell\pi}\left[\cos(\ell\pi)-\cos\left(\frac{\ell\pi}{2}\right)\right]$$

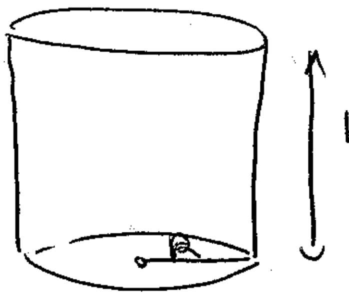
$$\longrightarrow \quad \alpha_{nm\ell} = 8\cdot\left(\frac{1}{n\pi}\right)\left(\frac{1}{m\pi}\right)\left(1-\cos(n\pi)\right)\left(1-\cos(m\pi)\right)$$ [3]

$$\cdot\left\{\frac{10}{\ell\pi}\left(1-\cos\left(\frac{\ell\pi}{2}\right)\right) + \frac{50}{\ell\pi}\left(\cos\left(\frac{\ell\pi}{2}\right)-\cos(\ell\pi)\right)\right\}$$

Global diffusion timescale: corresponds to λ_{111}:

$$\tau_{111} = \frac{1}{\lambda_{111}} = \frac{1}{3\pi^2}$$ [2]

Problem 2 : [25]



$V = 1 \iff \pi R^2 h = 1$

$\iff h = \frac{1}{\pi R^2}$ [1]

We solve the same equation, $\frac{\partial T}{\partial t} = \nabla^2 T$
but this time in cylindrical coordinates. $\nabla^2 T =$ see below

- $T(r, \theta, z, 0) = \begin{cases} 10 & \text{if } z < \frac{h}{2} \\ 50 & \text{if } z > \frac{h}{2} \end{cases}$ [3].
- $T = 0$ at $r = R$, $z = 0$, $z = h$.

Separation of variables: $T(r, \theta, z, t) = A(r, \theta, z) B(t)$

$$\begin{cases} B_t = -\lambda B \\ \nabla^2 A = -\lambda A \end{cases}$$

Note: In these solutions I forgot that the system is axisymmetric and did it in general case... But you may assume $\partial/\partial\theta = 0$ indeed

$$\nabla^2 A = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 A}{\partial \theta^2} + \frac{\partial^2 A}{\partial z^2} = -\lambda A$$

$\Rightarrow$ let $A(r, \theta, z) = \alpha(r)\beta(\theta)\gamma(z)$

$$\Rightarrow \frac{\beta\gamma}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \alpha}{\partial r}\right) + \frac{1}{r^2}\alpha\gamma\frac{\partial^2 \beta}{\partial \theta^2} + \alpha\beta\frac{\partial^2 \gamma}{\partial z^2} = -\lambda A$$

$$\Rightarrow \frac{1}{\alpha}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \alpha}{\partial r}\right) + \frac{1}{r^2}\frac{1}{\beta}\frac{\partial^2 \beta}{\partial \theta^2} + \frac{1}{\gamma}\frac{\partial^2 \gamma}{\partial z^2} = -\lambda$$

$$\Rightarrow \frac{1}{\alpha}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \alpha}{\partial r}\right) + \frac{1}{r^2}\frac{1}{\beta}\frac{d^2 \beta}{d\theta^2} = \underbrace{-\frac{1}{\gamma}\frac{d^2 \gamma}{dz^2} - \lambda}_{\text{f. of } z \text{ only}} = -K$$

$$\frac{1}{\alpha}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \alpha}{\partial r}\right)+\frac{1}{r^2}\frac{1}{\beta}\frac{d^2\beta}{d\theta^2}=-k$$

$$\Rightarrow \quad \frac{r}{\alpha}\frac{d}{dr}\left(r\frac{d\alpha}{dr}\right)+kr^2 = -\frac{1}{\beta}\frac{d^2\beta}{d\theta^2} = C$$

$$\Rightarrow \begin{cases} \dfrac{d^2\beta}{d\theta^2} = -C\beta \\ r\dfrac{d}{dr}\left(r\dfrac{d\alpha}{dr}\right)+kr^2\alpha = C\alpha \\ \dfrac{d^2\gamma}{dz^2} = (-\lambda+k)\,\gamma \end{cases} \qquad [6]$$

- β has to be perodic $\rightarrow$ $C=n^2$ and

$$\beta_n(\theta) = a_n\cos n\theta + b_n \sin n\theta \qquad [2]$$

- $$r^2\frac{d^2\alpha}{dr^2}+r\frac{d\alpha}{dr}+kr^2\alpha - n^2\alpha = 0$$

$\rightarrow$ looks like a Bessel function equation provided $x^2 = kr^2 \qquad x=\sqrt{k}\,r$

$$\rightarrow \quad \alpha_{nm}(r) = J_n(\sqrt{k_{nm}}\,r) \qquad [2].$$

To satisfy bcs at $r=R$ then $\sqrt{k_{nm}}\,R$ = zero of Bessel J_n

$= z_{n,m}$ [2].

$$\rightarrow \quad \alpha_{nm}(r) = J_n\left(\frac{z_{nm}\,r}{R}\right)$$

- finally: $$\frac{d^2\gamma}{dz^2} = \left[-\lambda+\left(\frac{z_{nm}}{R}\right)^2\right]\gamma$$

$\rightarrow$ needs to satisfy the BCs so the number in brackets has to be $-\frac{\ell^2\pi^2}{h^2}$

[2]

[2] $\gamma(z) = \sin\left(\frac{\ell\pi z}{h}\right) \quad \Rightarrow -\lambda_{\ell mn}+\left(\frac{z_{nm}}{R}\right)^2 = -\frac{\ell^2\pi^2}{h^2}$

so $\lambda_{lmn} = \frac{l^2\pi^2}{h^2} + \left(\frac{z_{nm}}{R}\right)^2 \rightarrow \tau_{lmn} = (\lambda_{lmn})^{-1}$

Note: For axisymmetric case get $\lambda_{lm} = \frac{l^2\pi^2}{h^2} + \left(\frac{z_{0m}}{R}\right)^2$

The smallest possible value of λ_{lmn} occurs for $l=1$, and the minimum of z_{nm} for n, m spanning all possible values

nth zeros of	J_0	J_1	J_2
1	2.4	3.83	⋮
2	5.52	7.01	⋮

[2]

→ the lowest one is the first zero of J_0: 2.404....

→ λ_{lmn} is minimum for $l=1$, $n=0$, $m=1$ (first zero)

$$\lambda_{110} = \frac{\pi^2}{h^2} + \frac{(z_{01})^2}{R^2}$$

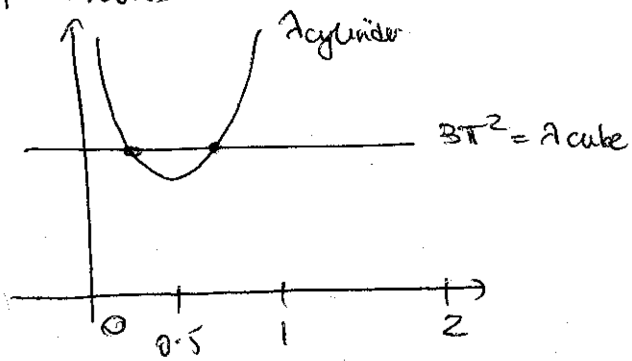
but $h = \frac{1}{\pi R^2}$ ⇒

$$\lambda_{110} = \frac{\pi^2}{\left(\frac{1}{\pi R^2}\right)^2} + \frac{(z_{01})^2}{R^2} = \pi^4 R^4 + \frac{(z_{01})^2}{R^2}$$

$\lambda_{cylinder} = \lambda_{cube}$ ⇔

$$\pi^4 R^4 + \left(\frac{z_{01}}{R}\right)^2 = 3\pi^2 \qquad [2]$$

Plot shows



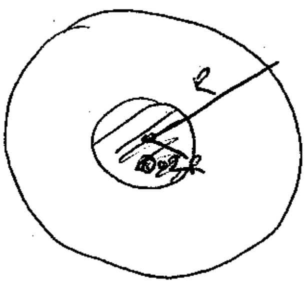
Intersections at

$R = 0.491 \rightarrow \boxed{\frac{h}{R} = 2.689}$

$R = 0.623 \rightarrow \boxed{\frac{h}{R} = 1.316}$

[3]

Problem 3 [25]



$$\begin{cases} \frac{\partial^2 p}{\partial t^2} = c_0^2 \left[\frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial p}{\partial r} \right) \right] \\ p(0.2R, t) = p(R,t) = 0 \\ c_0 = 4 10^6 \quad \text{cm/s.} \end{cases}$$

(a) Radial eigenmode satisfies

[2] $$\frac{c_0^2}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial p}{\partial r} \right) = -\omega^2 p \quad \text{for } r \in (0.2R, R)$$

expect this form of constant for oscillations

[1] This is a regular problem since $r=0$ is not in the domain of interest.

(b) If $c(r) = c_0$ then

$$\frac{d^2 p}{dr^2} + \frac{2}{r}\frac{dp}{dr} = -\frac{\omega^2}{c_0^2} p \Rightarrow r^2 \frac{d^2 p}{dr^2} + 2r\frac{dp}{dr} + \frac{\omega^2}{c_0^2} p r^2 = 0$$

[3] ↳ This is a spherical Bessel function equation for the 0th-order functions, provided $x = \frac{\omega^2}{c_0^2} r^2$ $x = \frac{\omega}{c_0} r$

[2] ↳ the solutions are $j_0(x)$ and $y_0(x)$, so the general solution for p is

[2]. $$p(r) = a\, j_0\left(\frac{\omega}{c_0} r\right) + b\, y_0\left(\frac{\omega}{c_0} r\right) \quad a, b \text{ to be determined}$$

(c) at $r = R$: $p(R) = 0$

[2] $$\Rightarrow a\, j_0\left(\frac{\omega}{c_0} R\right) + b\, y_0\left(\frac{\omega}{c_0} R\right) = 0$$

at $r = 0.2R$:

[2] $\Rightarrow \quad a\, j_0\left(\frac{\omega}{c_0}\cdot\frac{R}{5}\right) + b\, y_0\left(\frac{\omega}{c_0}\frac{R}{5}\right) = 0$

This is a system of 2 equations for 2 variables, of the type $MX = 0$ $\Rightarrow$ only has non-trivial solutions if $\det M = 0$ $\Rightarrow$

$$j_0\left(\frac{\omega}{c_0}R\right) y_0\left(\frac{\omega}{c_0}\frac{R}{5}\right) - j_0\left(\frac{\omega}{c_0}\frac{R}{5}\right) y_0\left(\frac{\omega}{c_0}R\right) = 0$$

Or, in other words, if $\xi = \frac{\omega R}{c_0}$, then

[2] $$f(\xi) = j_0(\xi)\, y_0\left(\frac{\xi}{5}\right) - j_0\left(\frac{\xi}{5}\right) y_0(\xi) = 0$$

(d) Plotting $f(\xi)$ yields the first 5 zeros

[5] approximately at $\xi = 4, 8, 12, 15.5$ and 19.5

(e) so the fundamental period of oscillation is:

[4] $$T_1 = \frac{2\pi}{\omega_1} \cong \frac{2\pi R}{4c_0} \simeq \frac{\pi}{2}\cdot 3\,10^3 = 4.71\,10^3 \text{ seconds} \simeq 1.3 \text{ hours}$$

$$T_2 = \frac{2\pi}{\omega_2} \cong \frac{2\pi R}{8c_0} \simeq \frac{T_1}{2} = 0.65 \text{ hours.}$$

<u>Note</u>: As it happens $j_0(\xi) = \frac{\sin\xi}{\xi}$ and $y_0(\xi) = -\frac{\cos\xi}{\xi}$

so $j_0(\xi)\, y_0\left(\frac{\xi}{5}\right) - j_0\left(\frac{\xi}{5}\right) y_0\left(\frac{\xi}{5}\right) = 0$

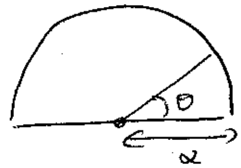
$\Leftrightarrow \sin(\xi)\cos\left(\frac{\xi}{5}\right) - \sin\left(\frac{\xi}{5}\right)\cos\left(\frac{\xi}{5}\right) = 0$

$\Leftrightarrow \sin\left(\frac{4\xi}{5}\right) = 0 \quad \Rightarrow \frac{4\xi}{5} = n\pi \Rightarrow \boxed{\xi = \frac{5n\pi}{4}}$

and so $\omega_n = \frac{c_0}{R}\cdot\frac{5n\pi}{4}$ $\quad T_0 = \frac{2\pi}{\omega_1}$

Problem 4 (Textbook 8.5-6) [25].

$$\begin{cases} \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2} = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} + g(r,\theta,t) \\ \dfrac{\partial u}{\partial r} = 0 \quad \text{at } r = \alpha \\ u(r,0,t) = u(r,\pi,t) = 0 \\ u(r,\theta,0) = H(r,\theta), \quad u_t(r,\theta,0) = 0 \end{cases}$$



① let's look for the homogeneous problem first.

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

assuming $u(r,\theta,t) = \sum a(t)\,\phi(r,\theta)$ we have

$$\frac{\phi}{c^2}\frac{d^2 a}{dt^2} = \frac{a}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{a}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

$$\rightarrow \quad \frac{1}{ac^2}\frac{d^2 a}{dt^2} = \frac{1}{\phi}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}\right] = -\lambda$$

$\rightarrow$ ϕ is the solution of $\nabla^2\phi = -\lambda\phi$
where $\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$

There will be an ∞ of such solutions, referenced by the indices n and m with $\nabla^2\phi_{nm} = -\lambda_{nm}\phi$;
By Sturm-Liouville theory the ϕ_{nm} are orthogonal,
with $\langle \phi_n, \phi_m \rangle = \iint_D \phi_n \phi_m \, d^2\underline{r} = \int_{r=0}^{\alpha}\int_{\theta=0}^{\pi} r\,dr\,d\theta \; \phi_n \phi_m = 0$ if $n \neq$

Let's now go back to the original forced problem, and still assume $u(r,\theta,t) = \sum_{n,m} a_{nm}(t)\,\phi_{nm}(r,\theta)$

$$\rightarrow \quad \frac{1}{c^2}\sum_{n,m}\phi_{nm}\frac{d^2 a_{nm}}{dt^2} = \sum_{n,m}\nabla^2\phi_{nm}\cdot a_{nm} + g(r,\theta,t)$$

$$= -\sum_{n,m}\lambda_{nm}\phi_{nm}\,a_{nm} + g(r,\theta,t)$$

so, by projecting onto the ϕ_{nm} we get

$$\frac{1}{c^2}\frac{d^2 a_{nm}}{dt^2} = -\lambda_{nm} a_{nm} + \frac{\int_0^{\alpha}\int_0^{\pi} r\,dr\,d\theta\; g(r,\theta,t)\,\phi_{nm}(r,\theta)}{\int_0^{\alpha}\int_0^{\pi} \phi_{nm}^2 \; r\,dr\,d\theta}.$$

[4] $$= -\lambda_{nm} a_{nm} + \frac{\langle g, \phi_{nm}\rangle}{\langle \phi_{nm}, \phi_{nm}\rangle}$$

At time $t=0$, $u(r,\theta,0) = H(r,\theta)$ so

$$H(r,\theta) = \sum_{nm} a_{nm}(0)\,\phi_{nm}(r,\theta)$$

[2] $$\rightarrow \quad a_{nm}(0) = \frac{\langle H, \phi_{nm}\rangle}{\langle \phi_{nm}, \phi_{nm}\rangle}$$

$$u_t(r,\theta,0) = 0 \quad \text{so}$$

[2] $$0 = \sum_{nm} a_{nm}'(0)\,\phi_{nm}(r,\theta) \Rightarrow a_{nm}'(0) = 0$$

(b) Let's now solve for ϕ_{nm} and the eigenvalues λ_{nm}.

let $\phi_{nm}(r,\theta) = A(r)B(\theta)$ then

$$\frac{B}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right) + \frac{A}{r^2}\frac{\partial^2 B}{\partial \theta^2} = -\lambda AB$$

$$\rightarrow \quad \frac{1}{Ar}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right) + \frac{1}{Br^2}\frac{\partial^2 B}{\partial \theta^2} = -\lambda$$

[3] $$\rightarrow \quad \frac{r}{A}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right) + \lambda r^2 = -\frac{1}{B}\frac{\partial^2 B}{\partial \theta^2} = +n^2$$

$$\rightarrow \frac{d^2 B}{d\theta^2} = -n^2 B \quad \rightarrow \quad B_n(\theta) = \begin{Bmatrix} \cos n\theta \\ \sin n\theta \end{Bmatrix}$$

note: n has to be an integer to satisfy ~~periodicity in θ~~.

Then $$\frac{r}{A}\frac{d}{dr}\left(r\frac{dA}{dr}\right) + \lambda r^2 - n^2 = 0$$

$$\Rightarrow \quad r^2\frac{d^2A}{dr^2} + r\frac{dA}{dr} + (\lambda r^2 - n^2)A = 0$$

[3] $$\rightarrow \quad A(r) = \begin{Bmatrix} J_n(\sqrt{\lambda}\,r) \\ Y_n(\sqrt{\lambda}\,r) \end{Bmatrix}$$ (see lecture notes)

To satisfy the boundary conditions in θ

$$u(r,0,t) = u(r,\pi,t) = 0$$

with $B_n(\theta) = \alpha_n \cos n\theta + \beta_n \sin n\theta$

we have: $\alpha_n = 0$

$\beta_n \sin n\pi = 0$ → works –

So $B_n(\theta) = \sin n\theta$

To satisfy the BCs in r:

regularity at $r=0$ ⇒ $Y_n(\sqrt{\lambda}\, r)$ solution must be discarded

$\frac{du}{dr} = 0$ at $r=0$ ⇒ $J_n'(\sqrt{\lambda}\,\alpha) = 0$

⇒ $\sqrt{\lambda}\,\alpha$ is a zero of $\frac{dJ_n}{dr}$.

Let's call it z_{nm}.

[2] So $\lambda_{nm} = \left(\frac{z_{nm}}{\alpha}\right)^2$

⇒ Finally, $\phi_{nm}(r,\theta) = \sin(n\theta) \cdot J_n\left(\frac{z_{nm}}{\alpha} r\right)$

where z_{nm} is the m^{th} zero of $\frac{dJ_n}{dr}$.

(c) The solution to $f'' = -\omega^2 f + g(t)$ being

$$f(t) = \frac{1}{\omega}\int_0^t g(t')\sin(\omega(t-t'))\,dt'$$

the solution to the time-dependent problem we have is

[4] $$a_{nm}(t) = \frac{1}{\sqrt{\lambda_{nm}}\,c}\int_0^t c^2 \frac{\langle g, \phi_{nm}\rangle}{\langle \phi_{nm}, \phi_{nm}\rangle} \cdot \sin\left(c\sqrt{\lambda_{nm}}\,(t-t')\right)dt'$$
$$+ C_1 \cos\left(\sqrt{\lambda_{nm}}\,ct\right) + C_2 \sin\left(\sqrt{\lambda_{nm}}\,ct\right)$$

C_1, C_2 integration const

To satisfy the initial conditions:

$$a_{nm}'(t) = \frac{c}{\sqrt{\lambda_{nm}}}\left[\cancel{\frac{\langle g, \phi_{nm}\rangle}{\langle \phi_{nm}\phi_{nm}\rangle} \sin\left(c\sqrt{\lambda_{nm}}(t-t)\right)}^{0} + \int_0^t \frac{\langle g\phi_{nm}\rangle}{\langle \phi_{nm}\phi_{nm}\rangle} \cos\left(c\sqrt{\lambda_{nm}}(t-t')\right) . c\sqrt{\lambda_{nm}} \, . \, dt' \right]$$

$$+\left(-c_1\sqrt{\lambda_{nm}}\,c\right)\sin\left(\sqrt{\lambda_{nm}}\,ct\right) + c_2\sqrt{\lambda_{nm}}\,c\cos\left(\sqrt{\lambda_{nm}}\,ct\right)$$

$$\Rightarrow a_{nm}'(0) = 0 \iff \boxed{c_2 = 0}$$

$$a_{nm}(0) = 0 \iff \cancel{\frac{c}{\sqrt{\lambda_{nm}}}\int_0^0 \frac{\langle g\phi_{nm}\rangle}{\langle \phi_{nm}\phi_{nm}\rangle} \sin\left(\sqrt{\lambda_{nm}}\,c\,(0-t')\right) dt'}^{0}$$

[5]

$$+ c_1 = \frac{\langle H\phi_{nm}\rangle}{\langle \phi_{nm}\phi_{nm}\rangle} \Rightarrow c_1 = \frac{\langle H\phi_{nm}\rangle}{\langle \phi_{nm}\phi_{nm}\rangle}$$

So finally:

$$u(r,\theta,t) = \sum_{n,m}{}' \sin(n\theta) . J_n\left(\frac{z_{nm}}{a}r\right) . \left\{ \frac{\langle H\phi_{nm}\rangle}{\langle \phi_{nm}\phi_{nm}\rangle} . \cos\left(\sqrt{\lambda_{nm}}\,ct\right) + \frac{c}{\sqrt{\lambda_{nm}}} \int_0^t dt' \frac{\langle g\,\phi_{nm}\rangle}{\langle \phi_{nm}\,\phi_{nm}\rangle} \sin\left(\sqrt{\lambda_{nm}}\,c(t-t')\right) \right.$$