AMS 212A Final 2012

There are 4 problems on this final. Each is worth 25 points. Any substantially completed answer gives you 5 points extra credit, so that answering 3 problems perfectly is sufficient for an A.

Calculators are not allowed. You need to justify all your answers. Answers without justifications will be counted as wrong. If you have any hesitation about the question, if you think there may be a problem with it or if you just need clarification, DON'T HESITATE TO ASK!

Problem 1: Greens' function for the damped wave equation

A more realistic description of forced vibrations is given by the damped wave equation:

$$u_{tt} + Du_t = c^2 u_{xx} + F(x,t).$$

Here we consider a simple system with $u(x,0) = u_t(x,0) = 0$, and $u(0,t) = u(L,t) = 0$. This problem walks you through the construction of the Green's function for this equation.

(a) Find the eigensolutions (eigenvalues λ_n and eigenfunctions $v_n(x)$) of the associate homogeneous spatial eigenvalue problem $c^2 v_{xx} = -\lambda v$.

(b) Given $u(x,t) = \sum_n a_n(t) v_n(x)$, find the ODEs and initial conditions satisfied by the functions $a_n(t)$.

(c) Given that the inverse Laplace transform of $\frac{\hat{f}(s)}{(s+a)(s+b)}$ is, for any a and b, given by

$$\frac{1}{a-b}\int_0^t f(t')\left(e^{b(t'-t)} - e^{a(t'-t)}\right) dt'$$

where $\hat{f}(s)$ is the Laplace transform of f, find the solutions to these ODEs.

(d) Using your result, write down the formal solution of the problem, in the form of an integral of $F(x,t)$ times a Green's function. What is the Green's function for this damped oscillation problem?

Problem 2: Temperature in a disk

What is the temporal evolution of the temperature profile inside a flat disk of radius $R = 1$ (you may assume it is 2D), under the following model:

$$T_t = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}$$
$$T(r,\theta,0) = 0$$
$$T(R,\theta,t) = 100 \text{ for } t > 0$$

Your solution needs to be completely explicit (i.e. evaluate all integrals that may arise from your calculations).

Problem 3: Spherical Green's functions

This problem aims to calculate the Green's function for the spherically symmetric Poisson equation in the spherical cavity $r \in [a, b]$ (where a and b are non-zero) in two different ways. Note that this is an ODE problem.

Consider the Poisson problem

$$L(u) = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{du}{dr}\right) = F(r) \text{ for } r \in (a, b),$$
$$u(a) = u(b) = 0$$

(a) Show from first principles (i.e. not by plugging the solution in) that the eigenfunctions of the associate eigenproblem $L(v) = -\lambda v$ are

$$v_n(r) = \frac{1}{r}\sin\left(\frac{n\pi(r-a)}{b-a}\right)$$

What are the eigenvalues? Hint: you may have to use the fact that $j_0(x) = \sin(x)/x$ and $y_0(x) = \cos(x)/x$ (see Problem 2).

(b) Deduce what the Green's function for this problem is using eigenfunction expansions. All remaining integrals must be evaluated.

(c) Find the general solution of the equation $L(u) = 0$.

(d) What is the solution to $L(u) = \delta(r - r')$, under the same boundary conditions as the orginal problem to deduce what the Green's function is using the $\delta-$forcing.

Problem 4: Canonical forms

Consider the equation

$$u_{xx} + 4u_{xy} + u_x = 0$$

- Is this PDE hyperbolic, parabolic or elliptic?
- Show that the canonical form of the equation is $u_{\xi\eta} + \frac{1}{4}u_\eta = 0$
- Derive the general solution $u(x, y)$ and show that it can be written as $u(x, y) = F(y-4x)e^{-y/4}+G(y)$
- Find the specific solution satisfying the initial conditions

$$u(x, 8x) = 0 \text{ and } u_x(x, 8x) = 4e^{-2x}$$

You may need to use some of the following facts:

- The Bessel equation is $x^2 f'' + xf' + (x^2 - n^2)f = 0$. It has a regular solution $J_n(x)$ and a singular solution (at $x = 0$) $Y_n(x)$.
- The Spherical Bessel equation is $x^2 f'' + 2xf' + (x^2 - n(n+1))f = 0$. It has a regular solution $j_n(x)$ and a singular solution (at $x = 0$) $y_n(x)$.
- The Legendre equation is $(1 - x^2)f'' - 2xf' + n(n+1)f = 0$ and has a regular solution $P_n(x)$ and a singular solution (at $x = \pm 1$) $Q_n(x)$.