AMS 212A Final 2011

There are 4 problems on this final. Each is worth 30 points, so that answering 3 problems perfectly is sufficient for an A.

You need to justify all your answers. Answers without justifications will be counted as wrong. If you have any hesitation about the question, if you think there may be a problem with it or if you just need clarification, DON'T HESITATE TO ASK!

Calculators are not allowed.

Problem 1: Laplace's equation in a rectangle with non-homogeneous boundary conditions

Consider the rectangle defined by $x \in [0, L]$ and $y \in [0, H]$. We would like to solve the following problem

$$\nabla^2 u = 0$$
$$u(0, y) = a_1(y) \quad , \quad u(L, y) = a_2(y) \quad , \quad u(x, 0) = a_3(x) \quad , \quad u(x, H) = a_4(x) \tag{1}$$

- Verify that the solution to this problem can be written as $u(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + u_4(x, y)$ where

$$\nabla^2 u_1 = 0$$
$$u_1(0, y) = a_1(y) \quad , \quad u_1(L, y) = 0 \quad , \quad u_1(x, 0) = 0 \quad , \quad u_1(x, H) = 0 \tag{2}$$

  and where u_2, u_3 and u_4 satisfy a related set of boundary conditions which you must determine.

- What is the solution for u_1? You should express it as an infinite sum, which will contain terms involving integrals of $a_1(y)$.

- Either by direct calculation, or using symmetries of the problem, find the solutions for u_2, u_3 and u_4, and deduce the general formal solution for $u(x, y)$.

Problem 2:

The equation governing the diffusion of a magnetic field in a star is the following:

$$\frac{\partial B}{\partial t} = D \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial B}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial B}{\partial \theta} \right) - \frac{B}{r^2 \sin^2\theta} \right] \tag{3}$$

when expressed in the standard (r, θ, ϕ) spherical coordinate system. The boundary conditions are regular at $r = 0$, and have $B = 0$ at the stellar surface.

The star initially has a magnetic field with the following configuration:

$$B(r, \theta, 0) = B_0(r) \sin\theta \tag{4}$$

where $B_0(r)$ is an arbitrary function satisfying regularity at the origin, and $B_0(R) = 0$.

- Using separation of variables, assume that $B(r,\theta,t) = a(t)b(r)c(\theta)$ and find the 3 coupled ODEs satisfied by the functions a, b and c.
- Show that the angular eigenvalue equation has solutions of the kind

$$c_n(x) = P_n^1(x)$$

  with eigenvalue $K_n = n(n+1)$, where $x = \cos\theta$ and where the $P_n^m(x)$ are the Associated Legendre Functions (see textbook equation 7.10.13 and 7.10.15).
- Show that the radial eigenvalue equation has solutions of the kind

$$b_{mn}(r) = j_n(\lambda_{mn} r)$$

  where the j_n function is the spherical Bessel function of the first kind, and where you need to find the eigenvalue λ_{mn}. Note that the spherical Bessel functions $j_n(z)$ and $y_n(z)$ respectively are the regular and singular solutions of

$$z^2 f'' + 2zf' + (z^2 - n(n+1))f = 0 \tag{5}$$

  for integer values of n. The functions $j_n(z)$ are very similar to the regular Bessel functions $J_n(z)$ (see Abramowitz & Stegun page 438, Figure 10.1 for a graph).
- Using a similar method to the one proving the orthogonality of eigenfunctions of a regular Sturm-Liouville problem, show directly that

$$\int_0^R r^2 b_{mn}(r) b_{m'n}(r) dr = 0$$

  when $m \neq m'$. Hint: you will need to use the regularity condition at the origin.
- Solve the temporal equation, and write down the general solution $B(r,\theta,t)$ (your expression should contain a certain number of integration constants, that will be determined in the following two questions).
- Use the following formulas to show that only the $n = 1$ terms need to be kept:

$$P_n^m(x) = (1-x^2)^{m/2}\frac{d^m P_n}{dx^m} \quad , \quad P_0(x) = 1 \quad , \quad P_1(x) = x \quad , \quad P_2(x) = \frac{1}{2}(3x^2 - 1)$$

  (Note that the $P_n(x)$ functions are the Legendre Polynomials).
- Finally, fit the initial conditions to your simplified general solution to find the actual solution of the problem. Your solution will involve some integrals that you do not need to evaluate.

Problem 3: Canonical forms

Consider the equation and initial conditions:

$$u_{xx} - u_{xy} - 6u_{yy} = 0$$

- Is this equation hyperbolic, parabolic or elliptic?
- Cast the equation into canonical form
- Solve the equation to find the general solution
- Find the solution given the initial conditions $u(x,0) = f(x)$ and $u_y(x,0) = g(x)$.

Problem 4: Traffic flow

Consider the traffic flow problem

$$u_t + (uv(u))_x = 0$$
$$v(u) = 1 - u \text{ if } 0 < u < 1\,, v(u) = 0 \text{ otherwise}$$
$$u(x,0) = a \text{ if } x < 0$$
$$u(x,0) = a + (b-a)x \text{ if } x \in [0,1]$$
$$u(x,0) = b \text{ if } x > 1$$

- What is the equation for the characteristics of this problem? What is the value of u on these characteristics?

Now assume that $a = 1/2$ and $b = 0$.

- Draw the characteristics. What are the equations of the "transition characteristics" (i.e. the characteristics delimiting the various regions).
- Find the solution for $u(x,t)$, and sketch $u(x,t)$ for $t = 0$ and for a time $t > 0$. Annotate the axes carefully.

Now assume that $a = 1/2$ and $b = 3/4$.

- The characteristics in the intermediate region all cross in a single point. Show this.
- Draw the characteristics in all three regions. What are the equations of the "transition characteristics" (i.e. the characteristics delimiting the various regions)?. What kind of shock do you expect? Where does it start?
- Find the solution for $u(x,t)$ before and after the start of the shock.
- Draw a clean complete sketch with all characteristics and shocks.
- Sketch $u(x,t)$, for $t = 0$, for a time t before the start of the shock and for a time t after the start of the shock. Annotate the axes carefully.