Problem 1 $\quad u_{tt} + Du_t = c^2 u_{xx} + F(x,t)$

[7] (a) $\quad c^2 v_{xx} = -\lambda v$

$\rightarrow \quad v_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ to satisfy $u(0,t) = u(L,t) = 0$

$\rightarrow v_n(0) = v_n(L) = 0$

$\rightarrow \quad -\frac{n^2\pi^2c^2}{L^2} = -\lambda_n$

so $\lambda_n = \frac{n^2\pi^2c^2}{L^2}$

[7] (b) $\quad u(x,t) = \sum_n a_n(t)\, v_n(x)$

$$\Rightarrow \quad \ddot{a}_n + D\dot{a}_n = \left(-\frac{n^2\pi^2c^2}{L^2}\right) a_n + \frac{\int_0^L F(x,t)\, v_n(x)\, dx}{\int_0^L v_n^2(x)\, dx}$$

$$\Rightarrow \quad \ddot{a}_n + D\dot{a}_n + \frac{n^2\pi^2c^2}{L^2} a_n = f_n(t) \qquad [5]$$

where $f_n(t) = \frac{2}{L}\int_0^L F(x,t)\, v_n(x)\, dx$

At $t=0$ $\quad u(x,t) = 0 \Rightarrow a_n(0) = 0$

$t=0$ $\quad u_t(x,t) = 0 \Rightarrow \dot{a}_n(0) = 0$ $\qquad$ [2]

(c) Taking the Laplace Transform of the ODE

$$\rightarrow \quad s^2 \hat{a}_n - \cancel{s a_n(0)} - \cancel{\dot{a}_n(0)} + D\left(s\hat{a}_n - \cancel{a_n(0)}\right) + \frac{n^2\pi^2c^2}{L^2}\hat{a}_n = \hat{f}_n(s) \qquad [2]$$

$$\rightarrow \quad \hat{a}_n\left(s^2 + Ds + \frac{n^2\pi^2c^2}{L^2}\right) = \hat{f}_n(s)$$

$$\Rightarrow \quad \hat{a}_n = \frac{\hat{f}_n(s)}{s^2 + Ds + \frac{n^2\pi^2c^2}{L^2}} \qquad [3]$$

The denominator can be factored into $\dfrac{\hat{f}_n(s)}{(s+\alpha_n)(s+\beta_n)}$

with
$$\left.\begin{aligned}\alpha_n &= -\left(\frac{-D+\sqrt{D^2-\frac{4n^2\pi^2c^2}{L^2}}}{2}\right) = \frac{D-\sqrt{D^2-\frac{4n^2\pi^2c^2}{L^2}}}{2}\\ \beta_n &= -\left(\frac{-D-\sqrt{D^2-\frac{4n^2\pi^2c^2}{L^2}}}{2}\right) = \frac{D+\sqrt{D^2-\frac{4n^2\pi^2c^2}{L^2}}}{2}\end{aligned}\right)[4].$$

$\Rightarrow$ The function $a_n(t)$ is

$$a_n(t) = \frac{1}{\alpha_n-\beta_n}\cdot\int_0^t f_n(t')\left(e^{\beta_n(t'-t)}-e^{\alpha_n(t'-t)}\right)dt'$$

$$= \frac{1}{\sqrt{D^2-\frac{4n^2\pi^2c^2}{L^2}}}\int_0^t f_n(t')\left(e^{\alpha_n(t'-t)}-e^{\beta_n(t'-t)}\right)dt' \quad [1].$$

[6] (d) So $$u(x,t) = \sum_n \sin\left(\frac{n\pi x}{L}\right)\cdot\frac{1}{\sqrt{D^2-\frac{4n^2\pi^2c^2}{L^2}}}\int_0^t f_n(t')\left(e^{\alpha_n(t'-t)}-e^{\beta_n(t'-t)}\right)dt'$$

$$= \int_0^t\int_0^L F(x',t')\,G(x,x';t,t')\,dx'\,dt' \quad [1].$$

where

$$G(x,x';t,t') = \sum_n \frac{2}{L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x'}{L}\right)\frac{e^{\alpha_n(t'-t)}-e^{\beta_n(t'-t)}}{\sqrt{D^2-\frac{4n^2\pi^2c^2}{L^2}}} \quad [5].$$

Problem 2

$$\begin{cases} T_t = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} \\ T(r,\theta,0) = 0 \\ T(R,\theta,t) = 100 \end{cases}$$

+2 for noticing

$\Rightarrow$ Clear independence of problem in θ

so T is a function of (r,t) only

let $T(r,t) = A(r)B(t)$

$$AB_t = \frac{B}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right)$$

[5] $\rightarrow$ $$\frac{B_t}{B} = \frac{1}{A}\frac{1}{r}\frac{d}{dr}\left(r\frac{dA}{dr}\right) = -\lambda$$ (diffusion problem so expect decaying ($\lambda \geq 0$) exponential)

[10] = Two cases: $\lambda = 0$: $B =$ constant

$$\frac{d}{dr}\left(\frac{r\,dA}{dr}\right) = 0 \Rightarrow \frac{r\,dA}{dr} = K_1 \Rightarrow \frac{dA}{dr} = \frac{K_1}{r}$$

$$\Rightarrow A = +K_1 \ln r + K_2$$

[5] For regularity, we need $K_1 = 0 \rightarrow A = K_2$

This makes sense: at steady state, we expect entire disk to be at temperature $T = 100$

$\lambda \neq 0$: $B_t = -\lambda B$ so $B \sim e^{-\lambda t}$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dA}{dr}\right) = -\lambda A$$

$$\rightarrow r^2\frac{d^2A}{dr^2} + r\frac{dA}{dr} + \lambda r^2 A = 0$$

$$\rightarrow x^2\frac{d^2A}{dx^2} + x\frac{dA}{dx} + x^2 A = 0 \quad \text{if } x^2 = \lambda r^2$$

$\rightarrow$ a Bessel eq of order 0

[5] $$A_n(r) = J_0(\sqrt{\lambda_n}\, r)$$ λ_n to be determined

[5] So: combining everything we have

$$T(r,t) = T_0 + \sum_n a_n J_0(\sqrt{\lambda_n}\, r)\, e^{-\lambda_n t}$$

formerly "K_2"

As $t \to +\infty$ $T(r,t) = T_0$

We need this solution to have $T(R,t) = 100 \to T_0 = 100$

[2] So $T(r,t) = 100 + \sum_n{}' \alpha_n J_0(\sqrt{\lambda_n}\, r) e^{-\lambda_n t}$

$T(R,t) = 100 \Rightarrow \sum_n{}' \alpha_n J_0(\sqrt{\lambda_n}\, R) e^{-\lambda_n t} = 0 \ \forall t$

$\to J_0(\sqrt{\lambda_n}\, R) = 0 \ \forall t$ but $R = 1$

so in fact $\lambda_n = z_{0n}^2$ where z_{0n}

[3] is the nth zero of J_0.

$\Rightarrow T(r,t) = 100 + \sum_n \alpha_n J_0(z_{0n} r) e^{-z_{0n}^2 t}$

Finally, at $t=0$ we need $T(r,0) = 0 \Rightarrow$

$\sum_n{}' \alpha_n J_0(z_{0n} r) e^{-z_{0n}^2 \cdot 0} = \sum_{\alpha_n} \alpha_n J_0(z_{0n} r) = -100$

By orthogonality of eigenfunctions $\int_0^1 r J_0(z_{0n} r) J_0(z_{0m} r)\, dr = 0$ if $n \neq m$

[5] So $\alpha_n = \dfrac{-\int_0^1 100 J_0(z_{0n} r)\, r\, dr}{\int_0^1 J_0^2(z_{0n} r)\, r\, dr}$

So finally

$$T(r,t) = 100 \left[1 - \sum_n \underbrace{\frac{\int_0^1 r' J_0(z_{0n} r')\, dr'}{\int_0^1 r' J_0^2(z_{0n} r')\, dr'}}_{\text{related to } \alpha_n} \cdot J_0(z_{0n} r) e^{-z_{0n}^2 t}\right]$$

Problem 3

$$\begin{cases} \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) = F(r) & r \in (a,b) \\ u(a) = u(b) = 0 \end{cases}$$

[9] (a) $\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dv}{dr}\right) = -\lambda r$

$$\Rightarrow \quad r^2 \frac{d^2 v}{dr^2} + 2r\frac{dv}{dr} + \lambda r^2 v = 0$$

$$\Rightarrow \quad x^2 \frac{d^2 v}{dx^2} + 2x\frac{dv}{dx} + x^2 v = 0$$

$\hookrightarrow$ v is a linear combination of j_0 and y_0 functions

[3] $\Rightarrow \quad V(r) = \alpha j_0(\sqrt{\lambda}\, r) + \beta y_0(\sqrt{\lambda}\, r)$

$V(a) = V(b) = 0$

$$\Rightarrow \begin{cases} \alpha j_0(\sqrt{\lambda}\, a) + \beta y_0(\sqrt{\lambda}\, a) = 0 \\ \alpha j_0(\sqrt{\lambda}\, b) + \beta y_0(\sqrt{\lambda}\, b) = 0 \end{cases}$$

$$\Rightarrow \quad j_0(\sqrt{\lambda}\, a) y_0(\sqrt{\lambda}\, b) - j_0(\sqrt{\lambda}\, b) y_0(\sqrt{\lambda}\, a) = 0$$

$$\Rightarrow \quad \sin(\sqrt{\lambda}\, a)\cos(\sqrt{\lambda}\, b) - \cos(\sqrt{\lambda}\, a)\sin(\sqrt{\lambda}\, b) = 0$$

$$\Rightarrow \quad \sin(\sqrt{\lambda}\,(a-b)) = 0 \Rightarrow \sqrt{\lambda_n} = \frac{n\pi}{|(a-b)|}$$

[3] $\Rightarrow \lambda_n = \frac{n^2\pi^2}{(b-a)^2}$

Also: $\alpha \sin(\sqrt{\lambda}\, a) + \beta \cos(\sqrt{\lambda}\, a) = 0$

$$\Rightarrow \quad \frac{\beta}{\alpha} = -\tan(\sqrt{\lambda}\, a) = -\frac{\sin(\sqrt{\lambda}\, a)}{\cos(\sqrt{\lambda}\, a)}$$

so $V(r) = \alpha\left[\frac{\sin(\sqrt{\lambda}\, r)}{\sqrt{\lambda}\, r} - \frac{\sin(\sqrt{\lambda}\, a)}{\cos(\sqrt{\lambda}\, a)} \cdot \frac{\cos(\sqrt{\lambda}\, r)}{\sqrt{\lambda}\, r}\right]$

$$= K.\frac{1}{r}\left[\sin(\sqrt{\lambda}\, r)\cos(\sqrt{\lambda}\, a) - \sin(\sqrt{\lambda}\, a).\cos(\sqrt{\lambda}\, r)\right]$$

[3]. $= \frac{K}{r}\left[\sin(\sqrt{\lambda}.(r-a))\right] = \frac{K}{r}\sin\left(\frac{n\pi(r-a)}{(b-a)}\right)$

[6] (b) If $u(r) = \sum' c_n v_n(r)$ then

$$L(u) = \sum_n' -\lambda_n c_n v_n(r) = F(r)$$

But $\int_a^b r^2 v_n(r) v_m(r)\, dr = 0$ if $n \neq m$ so

$$\int_a^b \sum_n' -\lambda_n c_n v_n(r) v_m(r) r^2 dr = \int_a^b r^2 F(r) v_m(r)\, dr$$

[3] $$\Rightarrow \quad c_n = \frac{\int_a^b r^2 F(r) v_n(r)\, dr}{-\lambda_n \int_a^b r^2 v_n^2(r)\, dr}$$

So $$u(r) = \sum_n -\frac{1}{\lambda_n} \cdot \frac{\int_a^b r'^2 F(r') v_n(r')\, dr'}{\int_a^b r'^2 v_n^2(r')\, dr'} \cdot v_n(r)$$

[1] $$= \int_a^b G(r,r') F(r')\, dr' \quad \text{where}$$

$$G(r,r') = -\frac{1}{\lambda_n} \frac{r'^2 v_n(r') v_n(r)}{\int_a^b r''^2 v_n^2(r'')\, dr''}$$

$$= -\frac{1}{\lambda_n} \cdot \frac{r'}{r} \sin\left(\frac{n\pi(r-a)}{b-a}\right) \sin\left(\frac{n\pi(r'-a)}{b-a}\right) \cdot \frac{2}{(b-a}$$

[2]. since $$\int_a^b r'^2 v_n^2(r')\, dr' = \int_a^b \sin^2\left(\frac{n\pi(r'-a)}{b-a}\right) dr' = \frac{b}{2}$$

$$= -\frac{(b-a)^{\cancel{2}}}{n^2\pi^2} \frac{r'}{r} \sin\left(\frac{n\pi(r-a)}{b-a}\right) \sin\left(\frac{n\pi(r'-a)}{b-a}\right) \frac{2}{\cancel{b-a}}$$

$$= -\frac{2(b-a)}{n^2\pi^2} \frac{r'}{r} \sin\left(\frac{n\pi(r-a)}{b-a}\right) \sin\left(\frac{n\pi(r'-a)}{b-a}\right)$$

[5] (c) $L(u) = 0 \Rightarrow \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{du}{dr}\right) = 0$

$$\Rightarrow r^2 \frac{du}{dr} = A \Rightarrow \frac{du}{dr} = \frac{A}{r^2} \Rightarrow u = -\frac{A}{r} + B.$$

[10] (d) On the left: $u_L = -\frac{A_L}{r} + B_L$

$$u_R = -\frac{A_R}{r} + B_R$$

BC at $r=a$: $\underline{-\frac{A_L}{a} + B_L = 0}$ [1]

—— $r=b$: $\underline{-\frac{A_R}{b} + B_R = 0}$ [1]

Continuity: $\underline{-\frac{A_L}{r'} + B_L = -\frac{A_R}{r'} + B_R}$ [1].

Jump condition:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{du}{dr}\right) = \delta(r-r')$$

$$\rightarrow \int_{r'-\epsilon}^{r'+\epsilon} \cancel{dr}\frac{d}{dr}\left(r^2 \frac{du}{dr}\right) = \int_{r'-\epsilon}^{r'+\epsilon} r^2 \delta(r-r')\,dr$$

$$\rightarrow \left. r^2 \frac{du}{dr}\right]_{r'-\epsilon}^{r'+\epsilon} = r'^2$$

$$\Rightarrow \left. \frac{du}{dr}\right]_{r'-\epsilon}^{r'+\epsilon} = 1 \qquad [2]$$

So since $\frac{du}{dr} = \frac{A}{r^2}$:

$$\frac{A_R}{r'^2} - \frac{A_L}{r'^2} = 1 \Rightarrow \underline{A_R - A_L = r'^2} \quad [1].$$

So $B_L = \frac{A_L}{a}$, $B_R = \frac{A_R}{b}$

So $-\frac{A_L}{r'} + \frac{A_L}{a} = -\frac{A_R}{r'} + \frac{A_R}{b}$ [4].

$$\rightsquigarrow A_L = A_R\left(\frac{\frac{1}{b} - \frac{1}{r'}}{\frac{1}{a} - \frac{1}{r'}}\right)$$

$$\Rightarrow A_R\left(1 - \frac{\frac{1}{b} - \frac{1}{r'}}{\frac{1}{a} - \frac{1}{r'}}\right) = r'^2 \Rightarrow A_R = r'^2\left(\frac{\frac{1}{a} - \frac{1}{r'}}{\frac{1}{a} - \frac{1}{b}}\right) = \frac{r'^2}{\frac{r'}{b}}\left(\frac{r'-a}{b-a}\right)$$

$$\Rightarrow A_L = r'^2 \left(\frac{\frac{1}{b} - \frac{1}{r'}}{\frac{1}{a} - \frac{1}{b}} \right)$$

$$\Rightarrow B_L = \frac{r'^2}{a} \left(\frac{\frac{1}{b} - \frac{1}{r'}}{\frac{1}{a} - \frac{1}{b}} \right), \quad B_R = \frac{r'^2}{b} \left(\frac{\frac{1}{b} - \frac{1}{r'}}{\frac{1}{a} - \frac{1}{b}} \right)$$

So $$G(r, r') = \begin{cases} -\frac{A_L}{r} + B_L & r < r' \\ -\frac{A_R}{r} + B_R & r > r' \end{cases}$$

Problem 4 $\qquad u_{xx} + 4u_{xy} + u_x = 0$

[5] • $\delta = 2^2 - 4\cdot 0 = 4$

$\rightarrow$ a hyperbolic eq.

[10] • Consider $\dfrac{dy}{dx} = \dfrac{2 \pm \sqrt{\delta}}{1} = 2 \pm \sqrt{4} = \begin{cases} 4 \\ 0 \end{cases}$

$\dfrac{dy}{dx} = 4 \Rightarrow y = 4x + \xi \qquad$ so $\xi = y - 4x$ ✓ [3]

$\dfrac{dy}{dx} = 0 \Rightarrow y = \eta \qquad \eta = y$ [3]

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \cancel{\frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}} = -4\frac{\partial u}{\partial \xi}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}$$

$\Rightarrow \quad u_{xx} + 4u_{xy} + u_x = 0$

$$\rightarrow -4\frac{\partial}{\partial \xi}\left(-4\frac{\partial}{\partial \xi}\right)u + 4\cdot\left(-4\frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right)\right) + \left(-4\frac{\partial u}{\partial \xi}\right) = 0$$

$$\rightarrow \cancel{16u_{\xi\xi}} - \cancel{16u_{\xi\xi}} - 16u_{\xi\eta} - 4u_\xi = 0$$

$$\Rightarrow -16u_{\xi\eta} - 4u_\xi = 0$$

$$\Rightarrow u_{\xi\eta} + \frac{1}{4}u_\xi = 0 \qquad [4].$$

Not quite in the required form, but let's just swith ξ & η and we're there... So $\begin{cases} \xi = y \\ \eta = y - 4x \end{cases}$

to have $u_{\xi\eta} + \frac{1}{4}u_\eta = 0$

- $u_{\xi\eta} + \frac{1}{4} u_\eta = 0$

$$\to \quad u_\xi + \frac{1}{4} u = f(\xi) \qquad [3]$$

$$\to \quad \left(e^{\frac{1}{4}\xi} u\right)_\xi = e^{\frac{1}{4}\xi} f(\xi)$$

$$\to \quad e^{\frac{1}{4}\xi} u = F(\xi) + G(\eta) \qquad [3]$$

$$\to \quad u = \underbrace{e^{-\frac{1}{4}\xi} F(\xi)} + e^{-\frac{1}{4}\xi} G(\eta)$$

$$u = K(\xi) + e^{-\frac{1}{4}\xi} G(\eta)$$

[10] $$= K(y) + e^{-\frac{1}{4}y} G(y-4x) \quad \text{as required.}$$

[4].

- If we want $u(x, 8x) = 0$

$u_x(x, 8x) = 0$

[5] then $$0 = K(8x) + e^{-\frac{8x}{4}} G(8x-4x)$$

$$0 = K(8x) + e^{-2x} G(4x) \qquad [1]$$

$$u_x(x,y) = \cancel{\ }-4e^{-\frac{1}{4}y} G'(y-4x)$$

$$u_x(x,8x) = -4e^{-\frac{8x}{4}} G'(8x-4x) = -4e^{-2x} G'(4x) = 4e^{-2x}$$

[1].

$$\Rightarrow G'(4x) = -1$$

If $G'(u) = -1 \Rightarrow G(u) = -u + \text{const}.$ [1]

so $G(4x) = -4x + \text{const} = -4x + G_0$

so $K(8x) = -G(4x) e^{-2x} = (4x - G_0) e^{-2x}$

$$\to K(u) = \left(\frac{u}{2} - G_0\right) e^{-\frac{u}{4}} \qquad [1]$$

$$u(x,y) = \left(\frac{y}{2} - \cancel{G_0}\right) e^{-\frac{y}{4}} + e^{-\frac{y}{4}} \left(4x - y + \cancel{G_0}\right) \qquad [1].$$

$$= e^{-\frac{y}{4}} \left(4x - \frac{y}{2}\right) \longrightarrow \text{check:} \begin{cases} u_{xx} = 0 \quad u_x = 4e^{-y/4} \ \checkmark \\ u_{xy} = -e^{-y/4} \ \checkmark \\ u(x, 8x) = 0 \\ u_x(x, 8x) = 4e^{-\frac{8x}{4}} = 4e^{-2x} \ \checkmark \end{cases}$$