PROBLEM 1

$$\begin{cases} u_t + u u_x = 0 \\ u(x,0) = u_0(x) = \tanh\left(\frac{x}{\epsilon}\right) \end{cases}$$

- Initial condition curve: $t_0(s) = 0 \quad x_0(s) = s \quad u_0(s) = u_0(s)$.
- Characteristics:

$$\frac{dt}{d\tau} = 1 \quad \to \quad t = \tau + t_0(s) = \tau$$

$$\frac{dx}{d\tau} = u \quad \longrightarrow \quad \frac{dx}{d\tau} = u_0(s) \to x = u_0(s)\tau + s.$$

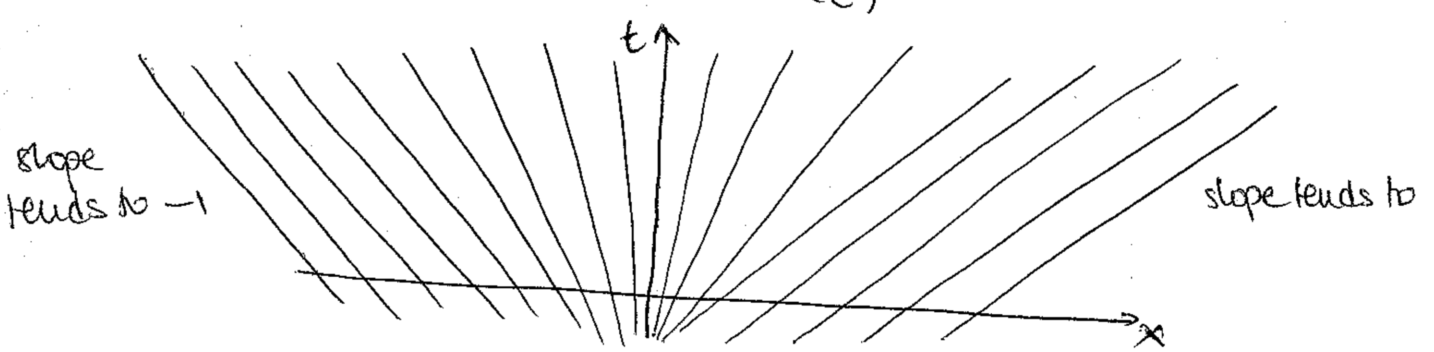
$$\frac{du}{d\tau} = 0 \quad \Rightarrow \quad u = u_0(s) = \text{constant}$$

$\Rightarrow \quad x = \tanh\left(\frac{s}{\epsilon}\right)t + s$ is the equation for the characteristics, in other word

$\Rightarrow \quad t = \dfrac{x - s}{\tanh\left(\frac{s}{\epsilon}\right)}$ → defined every where if $s \neq 0$.

If $s = 0$ then the characteristic is $x = 0$.

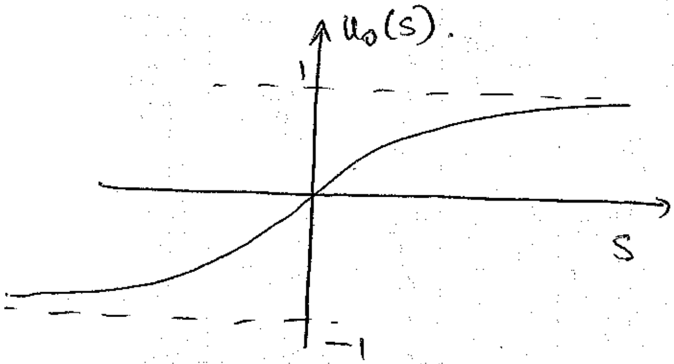
As $s \to -\infty$, $\tanh\left(\frac{s}{\epsilon}\right) \to -1$
$s \to +\infty$ $\tanh\left(\frac{s}{\epsilon}\right) \to +1$ } slope of characteristic,



- Implicit solution for $u(x,t)$:

$$u(x,t) = u_0(x - vt) = \tanh\left(\frac{x-vt}{\epsilon}\right)$$

- ϵ large



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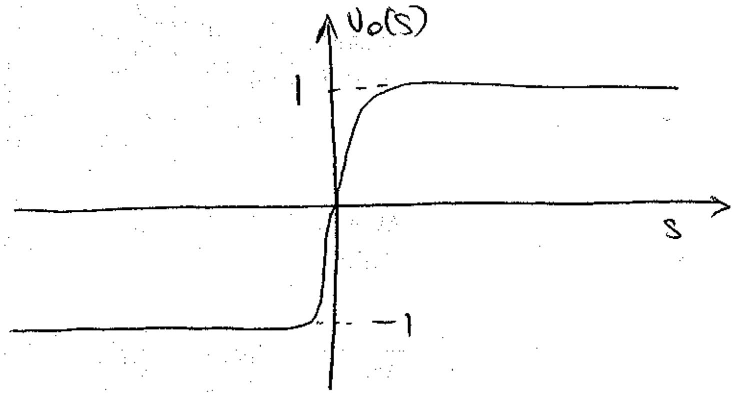


if $\epsilon \to 0$ then $\lim_{\epsilon\to 0} u_0(s) = \begin{cases} -1 & \text{if } s<0 \\ 1 & \text{if } s>0. \end{cases}$

- if $|s| \gg \epsilon$ so when $s \gg \epsilon$ $\quad u_0(s) \simeq +1 \Rightarrow u(x,t) = 1$

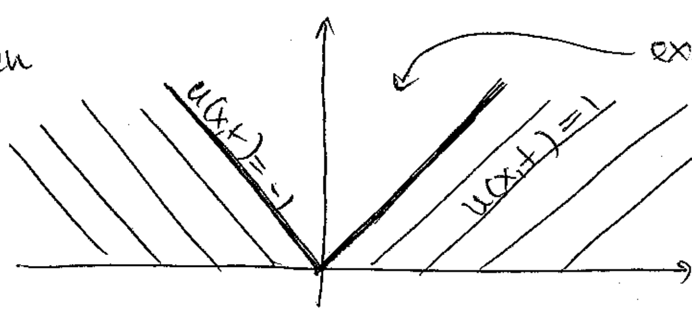
  $s \ll -\epsilon$ $\quad u_0(s) \simeq -1 \Rightarrow u(x,t) \simeq -$

- if $s \ll \epsilon$ then $u_0(s) \simeq \frac{s}{\epsilon}$ so we have

$$\begin{cases} x = \frac{s}{\epsilon}t + s \Rightarrow & s = \dfrac{x}{\frac{t}{\epsilon}+1} \\ u = \frac{s}{\epsilon} = \dfrac{x}{t+\epsilon} \simeq \dfrac{x}{t} \end{cases}$$

- If we had used instead $u_0(s) = \begin{cases} +1 & \text{if } s>0 \\ -1 & \text{if } s<0 \end{cases}$

then



expansion shock, where

$u(x,t) = G\left(\frac{x-0}{t}\right)$

G = inverse of $F'(u) = $

$\to u(x,t) = \frac{x}{t}$

as earlier.

$$4y^2 u_{xx} + 2(1-y^2)u_{xy} - u_{yy} - \frac{2y}{1+y^2}(2u_x - u_y) = 0$$

(a). $\delta = (1-y^2)^2 + 4y^2 = 1 - 2y^2 + y^4 + 4y^2 = 1+2y^2+y^4$

$= (1+y^2)^2$ Hyperbolic

$$\begin{cases} a = 4y^2 \\ b = (1-y^2) \\ c = -1 \end{cases}$$

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{4y^2}$$

$$= \frac{(1-y^2) \pm \sqrt{(1+y^2)^2}}{} = \begin{cases} 1-y^2 + (1+y^2) = 2 \\ 1-y^2 - (1+y^2) = -2y^2. \end{cases}$$

$$\frac{dy}{dx} = 2 \Rightarrow y = 2x + \xi \Rightarrow \boxed{\xi = y - 2x}$$

$$\frac{dy}{dx} = -2y^2 \Rightarrow -\frac{dy}{y^2} = 2dx \Rightarrow \frac{1}{y} = 2x + \eta$$

$$\boxed{\eta = \frac{1}{y} - 2x}$$

$$u_x = -2u_\xi + (-2)u_\eta = -2(u_\xi + u_\eta)$$

$$u_y = (1)u_\xi + \left(-\frac{1}{y^2}\right)u_\eta$$

$$u_{xx} = 4(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta})$$

$$u_{xy} = \frac{\partial}{\partial x}\left(u_\xi - \frac{1}{y^2}u_\eta\right)$$

$$= \frac{\partial}{\partial x}(u_\xi) - \frac{1}{y^2}\frac{\partial}{\partial x}(u_\eta) = -2(u_{\xi\xi} + u_{\eta\xi}) + \frac{2}{y^2}(u_{\xi\eta} + u_{\eta\eta})$$

$$u_{yy} = \frac{\partial}{\partial y}\left(u_\xi - \frac{1}{y^2}u_\eta\right) = \frac{\partial}{\partial y}(u_\xi) + \frac{2}{y^3}u_\eta - \frac{1}{y^2}\frac{\partial}{\partial y}(u_\eta)$$

$$= u_{\xi\xi} - \frac{1}{y^2}u_{\xi\eta} + \frac{2}{y^3}u_\eta - \frac{1}{y^2}\left(u_{\xi\eta} - \frac{1}{y^2}u_{\eta\eta}\right)$$

$$= u_{\xi\xi} - \frac{2}{y^2}u_{\xi\eta} + \frac{1}{y^4}u_{\eta\eta} + \frac{2}{y^3}u_\eta$$

PROBLEM 3

$$u_{tt} = c_s^2 \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2} \right]$$

(a) $u(a,z,t) = 0$ ← zero velocity at $r=a$

$u(r,0,t) = u(r,L,t) = 0$ ← zero velocity at $z=0,L$

$|u(r,z,t)| < +\infty$ ← regularity condition

(b) let $u(r,z,t) = A(r)B(z)C(t)$

$$AB\,C_{tt} = c_s^2 \left[\frac{BC}{r}\frac{\partial}{\partial r}\left(r \frac{\partial A}{\partial r}\right) + AC \frac{\partial^2 B}{\partial z^2} \right]$$

$$\rightarrow \frac{1}{C} C_{tt} = c_s^2 \left[\frac{1}{Ar}\frac{d}{dr}\left(r \frac{dA}{dr}\right) + \frac{1}{B}\frac{d^2 B}{dz^2} \right]$$

$=$ constant. Expect oscillations so let it be a negative constant

$= -\omega^2.$ $\Rightarrow$ $\boxed{C_{tt} = -\omega^2 C}$

$$\rightarrow \frac{1}{Ar}\frac{d}{dr}\left(r \frac{dA}{dr}\right) + \frac{1}{B}\frac{d^2 B}{dz^2} = -\frac{\omega^2}{c_s^2}$$

$$\rightarrow \frac{1}{Ar}\frac{d}{dr}\left(r \frac{dA}{dr}\right) + \frac{\omega^2}{c_s^2} = -\frac{1}{B}\frac{d^2 B}{dz^2} = \text{constant}.$$

$=$ Expect trapped modes with $B(0)=B(L)=0$
→ need oscillatory functions so let this constant be >0

$= k^2.$

$$\Rightarrow \quad \boxed{B_{zz} = -K^2 B}$$

So finally $\quad \frac{1}{Ar}\frac{d}{dr}\left(r\frac{dA}{dr}\right) = K^2 - \frac{\omega^2}{c_s^2}$

$$\rightarrow \quad \boxed{\frac{1}{r}\frac{d}{dr}\left(r\frac{dA}{dr}\right) = \left(K^2 - \frac{\omega^2}{c_s^2}\right)A}$$

(c) In the z-direction:

$$B(z) = \begin{Bmatrix} \cos Kz \\ \sin Kz \end{Bmatrix}$$

$B(0) = 0 \rightarrow$ no cosine

$B(L) = 0 \Rightarrow \sin(KL) = 0 \Rightarrow KL = n\pi \Rightarrow K = \frac{n\pi}{L}$

So we have n as the first index $\rightarrow$

z-eigenmodes are $B_n(z) = \sin\left(\frac{n\pi z}{L}\right)$

$$K_n = \frac{n\pi}{L}$$

(d) $\quad \frac{1}{r}\frac{d}{dr}\left(r\frac{dA}{dr}\right) = \left(\frac{n^2\pi^2}{L^2} - \frac{\omega^2}{c_s^2}\right)A$

Let's expand it $\rightarrow$

$$A_{rr} + \frac{A_r}{r} + \left(\frac{\omega^2}{c_s^2} - \frac{n^2\pi^2}{L^2}\right)A = 0$$

becomes

$$r^2 A_{rr} + rA_r + r^2\left(\frac{\omega^2}{c_s^2} - \frac{n^2\pi^2}{L^2}\right)A = 0$$

if $\frac{\omega^2}{c_s^2} - \frac{n^2\pi^2}{L^2} > 0$ then let

$$x^2 = r^2\left(\frac{\omega^2}{c_s^2} - \frac{n^2\pi^2}{L^2}\right)$$

$$x = r\sqrt{\frac{\omega^2}{c_s^2} - \frac{n^2\pi^2}{L^2}}$$

$$\rightarrow \quad x^2 A_{xx} + x A_x + x^2 = 0$$

$$\rightarrow A(x) = \begin{Bmatrix} J_0(x) \\ Y_0(x) \end{Bmatrix}.$$

(e) Since $Y_0(x)$ is singular, we discard it

$$\Rightarrow \quad A(r) = J_0\left(r\sqrt{\frac{\omega^2}{c_s^2} - \frac{n^2\pi^2}{L^2}}\right)$$

Applying $A(a) = 0 \Rightarrow$

$$a\sqrt{\frac{\omega^2}{c_s^2} - \frac{n^2\pi^2}{L^2}} = z_m$$

↑ mth 0 of the J_0 function.

$$\Rightarrow \quad \frac{\omega^2}{c_s^2} - \frac{n^2\pi^2}{L^2} = \frac{z_m^2}{a^2}.$$

$$\Rightarrow \quad \frac{\omega_{m,n}^2}{c_s^2} = \frac{n^2\pi^2}{L^2} + \frac{z_m^2}{a^2}$$ are the frequencies of the modes.

$$\Rightarrow \quad \omega_{mn} = c_s\sqrt{\frac{n^2\pi^2}{L^2} + \frac{z_m^2}{a^2}}.$$

And radial eigenmodes are $A_{mn}(r) = J_0\left(\frac{r z_m}{a}\right)$

(f) lowest frequency: $n=1$ and $z_1 \Rightarrow$

$$\omega_{11} = c_s\sqrt{\frac{\pi^2}{L^2} + \frac{z_1^2}{a^2}}$$

PROBLEM 4

$$T_t = ((1+x)^2 T_x)_x \quad , \quad T(0,t) = T(L,t) = 0$$
$$T(x,0) = T_0(x)$$

(a) • $y = 1+x$

$\Rightarrow \quad x = y-1 \qquad \frac{d}{dx} = \frac{d}{dy}$ so the equation easily becomes

$$T_t = (y^2 T_y)_y$$

• When $x=0 \quad y=1$ $\Rightarrow$ Boundary conditions become
$x=L \quad y=1+L \qquad T(1,t) = T(1+L,t) = 0.$

• $T(y,0) = T_0(\quad)$

(b) → see below

(c) $(y^2 u_y)_y + \lambda u = 0 \qquad u(1) = u(a) = 0$

Try a solution of the kind $u = y^\alpha$ then

$$\alpha(\alpha+1) + \lambda = 0 \quad \Rightarrow \quad \alpha^2 + \alpha + \lambda = 0$$
$$\Rightarrow \quad \alpha = \frac{-1 \pm \sqrt{1-4\lambda}}{2}.$$

$$\text{so } u(y) = A\, y^{\left(-\frac{1}{2} + \sqrt{1-4\lambda}\right)} + B\, y^{\left(-\frac{1}{2} - \sqrt{1-4\lambda}\right)}$$

to satisfy both bcs we will need oscillatory solutions → $1-4\lambda < 0$ ⇒ $u(y)$ also rewritten as

$$u(y) = y^{-1/2}\left[A\, y^{i\sqrt{4\lambda-1}} + B\, y^{-i\sqrt{4\lambda-1}}\right]$$
$$= y^{-\frac{1}{2}}\left[\hat{A}\cos\left(\sqrt{4\lambda-1}\,\ln y\right) + \hat{B}\sin\left(\sqrt{4\lambda-1}\,\ln y\right)\right].$$

since $y^\alpha = e^{\alpha \ln y}$

At $y=1$ need $u(y)=0$ $\Rightarrow$ cos term not needed

at $y=a$ need $u(a)=0$ $\Rightarrow$ $\sin\left(\sqrt{4\lambda-1}\,\ln a\right)=0$

$$\Rightarrow \sqrt{4\lambda-1}\,\ln a = n\pi$$

$$\Rightarrow 4\lambda_n - 1 = \frac{n^2\pi^2}{\ln^2 a}$$

$$\Rightarrow \lambda_n = \frac{1}{4}\left(\frac{n^2\pi^2}{(\ln a)^2}+1\right).$$

and $$V_n(y) = y^{-\frac{1}{2}} \sin\left(\frac{n\pi}{\ln a}\ln y\right)$$

(b) $$Q(u) = -\frac{\int_1^a u\,.\,(y^2 u_y)_y\,dy}{\int_1^a u^2\,dy}$$

$$= \frac{-\underbrace{\cancel{\left[u y^2 u_y\right]_1^a}}_{0 \text{ because of bcs.}} + \int_1^a y^2 u_y^2\,dy}{\int_1^a u^2\,dy} \geq 0$$

(d) Orthogonality:

$$\int_1^a V_n(y) V_m(y)\,dy = \delta_{mn}\|V_n\|^2$$

(e) $T_t = (y^2 T_y)_y$ $\rightarrow$ by separation of variables let

$$T(y,t) = A(y)B(t) \Rightarrow \frac{1}{B}B_t = \frac{1}{A}(y^2 A_y)_y = -\lambda$$

Then using previous questions $\begin{cases} B_n(t) = e^{-\lambda_n t} \\ A_n(y) = V_n(y) \end{cases}$ $\Rightarrow$ $T(y,t) = \sum_n B_n e^{-\lambda_n t} V_n(y)$

with V_n, λ_n as above with $a = 1+L$

$$T(y,t) = \sum_n{}' \beta_n e^{-\frac{1}{4}\left(\frac{n^2\pi^2}{[\ln(1+L)]^2}+1\right)t} y^{-1/2} \sin\left(\frac{n\pi}{(1+L)} \ln y\right).$$

$$\left[T(x,t) = \sum_n{}' \beta_n e^{-\frac{1}{4}\left(\frac{n^2\pi^2}{[\ln(1+L)]^2}+1\right)t} (1+x)^{-\frac{1}{2}} \sin\left(\frac{n\pi}{1+L} \ln(1+x)\right)\right]$$

Applying initial conditions we have

$T(x,0) = T_0(x)$ = as above. so $T(y,0) = T_0(y-1)$.

since $y = 1+x$

$\Rightarrow$ By orthogonality the β_n's are given by

$$\beta_n = \frac{\langle T_0(y-1), v_n(y)\rangle}{\langle v_n(y), v_n(y)\rangle}.$$