FINAL 2009 To be checked

Problem 1

① Let $v(x,t) = u(x,t) - h(x)f(t)$

If we want $v(0,t) = v(L,t) = 0$, since $u(0,t) = 0$ and $u(L,t) = f(t)$, we need:

$$v(0,t) = \cancel{u(0,t)}^{0} - h(0)f(t) = 0 \Rightarrow$$

$$\Rightarrow \quad h(0) = 0$$

$$v(L,t) = u(L,t) - h(L)f(t) = 0$$

$$= f(t) - h(L)f(t) = f(t)(1-h(L))$$

$$\Rightarrow \quad h(L) = 1$$

② The simplest function with this property is

$$h(x) = \frac{x}{L}$$

Then $v_t = u_t - \frac{x}{L}f'(t) = ku_{xx} - \frac{x}{L}f'(t)$

$$= k\left(v_{xx} - \cancel{h_{xx}}^{0} f(t)\right) - \frac{x}{L}f'(t)$$

$$= kv_{xx} - \frac{x}{L}f'(t) = kv_{xx} + F(x,t).$$

where $F(x,t) = -\frac{x}{L}f'(t)$

The BCs are automatically as required (by choice of h)

so $v(0,t) = v(L,t) = 0$

So :

$$\frac{d}{dt}\left(B_n \mu_n\right) = \mu_n \cdot f_n$$

$$\Rightarrow \quad B(t)\mu_n(t) = \int_0^t \mu_n f_n \, dt' + \mu_n(0) B_n(0)$$

$$\Rightarrow \quad B_n(t) = e^{-\frac{kn^2\pi^2}{L^2}t}\left[\int_0^t e^{\frac{kn^2\pi^2 t'}{L^2}} f_n(t')\,dt' + B_n(0)\right]$$

So :

$$V(x,t) = \sum_{n=1}^{\infty} A_n(x) B_n(t)$$

$$= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{kn^2\pi^2 t}{L^2}}\left[\int_0^t e^{\frac{kn^2\pi^2}{L^2}t'} f_n(t')\,dt' + B_n(0)\right]$$

<u>Initial Conditions</u>

If $\quad V(x,0) = V_0(x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) B_n(0)$

So $\quad B_n(0) = \frac{2}{L}\int_0^L V_0(x) \sin\left(\frac{n\pi x}{L}\right) dx.$

$\Rightarrow$ Finally :

$$V(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{kn^2\pi^2 t}{L^2}}\left[\int_0^t e^{\frac{kn^2\pi^2}{L^2}t'} f_n(t')\,dt' + \frac{2}{L}\int_0^L V_0(x') \sin\left(\frac{n\pi x'}{L}\right) dx'\right]$$

where f_n defined earlier

$$f_n(t) = \frac{2}{L}\int_0^L F(x,t) A_n(x)\,dx.$$

Finally, $v(x,0) = u(x,0) - \frac{x}{L} f(0)$

$$= u_0(x) - \frac{x}{L} f(0) = v_0(x)$$

(3) So now we have $v_t = k v_{xx} + F(x,t)$.

Let $v(x,t) = A(x)B(t)$

$$\hookrightarrow \begin{cases} B_t = -kB\lambda \\ A_{xx} = -\lambda A. \quad \text{with } A(0) = A(L) = 0 \end{cases}$$

Since the BCs in x are homogeneous, the solution is $A_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ with $\lambda = \frac{n^2\pi^2}{L^2}$

Let $$v(x,t) = \sum_{n=1}^{\infty} B_n(t) A_n(x)$$

$\Rightarrow$

$v_t = k v_{xx} + F(x,t)$ becomes

$$\sum_{n=1}^{\infty} \dot{B}_n(t) A_n(x) = \sum_{n=1}^{\infty} -k\frac{n^2\pi^2}{L^2} A_n(x) B_n(t) + F(x,t).$$

By orthogonality of the $A_n(x)$, we have

$$\dot{B}_n = -k\frac{n^2\pi^2}{L^2} B_n + \underbrace{\frac{\int_0^L F(x,t)A_n(x)\,dx}{\int_0^L A_n^2(x)\,dx}}_{\text{call this } f_n(t)}.$$

$\Rightarrow$ Solution: use integrating factor method (for ex.).

$$\mu_n(t) = e^{\int \frac{kn^2\pi^2}{L^2} dt} = e^{\frac{kn^2\pi^2}{L^2} t}$$

(4) If $u_0(x) = 0$ and $f(t) = 1 - e^{-t}$ then

- $V_0(x) = -\frac{x}{L} f(0) = -\frac{x}{L}$

- $F(x,t) = -\frac{x}{L} f'(t) = -\frac{x}{L} \cdot e^{-t}$

$$\Rightarrow \quad f_n(t) = \frac{2}{L} \int_0^L -\frac{x}{L} e^{-t} \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= -\frac{2}{L^2} e^{-t} \left\{ \left[-x \cos\left(\frac{n\pi x}{L}\right) \cdot \frac{L}{n\pi} \right]_0^L + \int_0^L \cos \frac{n\pi x}{L} \cdot \frac{L}{n\pi} \, dx \right\}$$

$$= -\frac{2}{L^2} e^{-t} \left\{ -\frac{L^2}{n\pi} \cos(n\pi) + \frac{L^2}{n^2\pi^2} \left[\sin\left(\frac{n\pi x}{L}\right) \right]_0^L \right\}$$

$$= +\frac{2}{\cancel{L^2}} e^{-t} \left\{ + \frac{\cancel{L^2}}{n\pi} (-1)^n \right\} = \frac{2}{n\pi} (-1)^n e^{-t}$$

$$V(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{kn^2\pi^2}{L^2} t} \left[\int_0^t e^{\frac{kn^2\pi^2}{L^2} t'} \cdot \frac{2}{n\pi} (-1)^n e^{-t'} \, dt' + \frac{2}{L} \int_0^L \left(-\frac{x'}{L} \right) \sin\left(\frac{n\pi x'}{L}\right) dx' \right].$$

$$= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{kn^2\pi^2}{L^2} t} \left[\frac{2}{n\pi} (-1)^n \cdot \frac{e^{\left(\frac{kn^2\pi^2}{L^2} - 1\right) t} - 1}{\frac{kn^2\pi^2}{L^2} - 1} + \frac{2}{n\pi} (-1)^n \right]$$

Problem 2 :

① Separation of variables: let $P(r,t) = A(r)\,B(t)$

$$\Rightarrow \begin{cases} B_{tt} = -\lambda B \\ \dfrac{c^2(r)}{r^2}\dfrac{d}{dr}\left(r^2 \dfrac{dA}{dr}\right) = -\lambda A \end{cases}$$

The spatial problem is a singular S.L problem with $p(r) = r^2$, $q(r) = 0$, $w(r) = \dfrac{r^2}{c^2(r)}$.

②
$$R(A) = \frac{-\int_0^R A \cdot \frac{d}{dr}\left(r^2 \frac{dA}{dr}\right) dr}{\int_0^R \frac{r^2}{c^2(r)} A^2(r)\,dr}$$

Using BCs

$$= \frac{\cancel{-\left[A \cdot r^2 \frac{dA}{dr}\right]_0^R}^{\to 0} + \int_0^R r^2 \left(\frac{dA}{dr}\right)^2 dr}{\int_0^R \frac{r^2}{c^2(r)} A^2(r)\,dr}$$

$$= \frac{\int_0^R r^2 \left(\frac{dA}{dr}\right)^2 dr}{\int_0^R \frac{r^2}{c^2(r)} A^2(r)\,dr} \geq 0 .$$

So λ are all positive.

③ let $c(r) = c_0$. Then we estimate λ_0 using

$\lambda_0 \simeq R(g)$ where $g(r)$ is a guess function with $g(R) = 0$ and $\left.\frac{dg}{dr}\right|_{r=0} = 0$

Let's try $g(r) = 1 - \left(\frac{r}{R}\right)^2$ $\qquad \frac{dg}{dr} = -\frac{2r}{R^2}$

$$\Rightarrow \lambda_0 \simeq \frac{\int_0^R r^2 \cdot \frac{4r^2}{R^4}\, dr}{\int_0^R \frac{r^2}{C_0^2}\left[1-\left(\frac{r}{R}\right)^2\right]^2 dr}$$

$$= \frac{\frac{4}{R^4}\left[\frac{r^5}{5}\right]_0^R}{\frac{1}{C_0^2}\left[\frac{r^3}{3} - \frac{2r^5}{5R^2} + \frac{r^7}{7R^4}\right]_0^R}$$

$$= \frac{\frac{4R}{5}}{\frac{1}{C_0^2}\left[\frac{R^3}{3} - \frac{2}{5}R^3 + \frac{R^3}{7}\right]} = \frac{\frac{4}{5}C_0^2}{R^2\left(\frac{35-42+15}{}\right)}$$

$$= \frac{C_0^2}{R^2} \cdot \frac{4}{5} \cdot \frac{105}{8} = \frac{C_0^2}{R^2} \cdot \frac{21}{2} = 10.5\,\frac{C_0^2}{R^2}$$

Since the frequency of the corresponding oscillation is $\omega_0^2 = \lambda_0 \Rightarrow \omega_0 = \sqrt{\lambda_0} = \sqrt{\frac{21}{2}}\,\frac{C_0}{R}$

then the fundamental period is $T_0 = \frac{2\pi}{\sqrt{\frac{21}{2}}\,\frac{C_0}{R}}$

$$\Rightarrow T_0 = \frac{2\pi}{\sqrt{\frac{21}{2}} \cdot \frac{10^7}{10^{11}}} \cong \frac{2\pi \cdot 10^4}{\sqrt{\frac{21}{2}}} \simeq 2\,10^4 \text{ s.}$$

(about 5 minutes)

(4) If $c(r) = C_0$ then the spatial problem becomes

$$\frac{d}{dr}\left(r^2 \frac{dA}{dr}\right) = -\lambda\,\frac{Ar^2}{C_0^2}$$

To make this look like a spherical Bessel equation,

e.g. $x^2 f'' + 2xf + (x^2 - n(n+1))f = 0$

$$\Rightarrow \quad r^2\frac{d^2A}{dr^2} + 2r\frac{dA}{dr} + \frac{\lambda}{C_0^2} r^2 A = 0$$

$\hookrightarrow$ need $n=0$, $\quad \frac{\lambda}{C_0^2} r^2 = x^2 \Rightarrow x = \frac{\sqrt{\lambda}}{C_0} r$

The solutions are therefore $j_0(x)$ and $y_0(x)$. To keep the regular ones only, discard y_0.

(5) Let's compute

$$x^2\left(\frac{\sin x}{x}\right)'' + 2x\left(\frac{\sin x}{x}\right)'$$

$$\left(\frac{\sin x}{x}\right)' = \frac{x\cos x - \sin x}{x^2} = \frac{\cos x}{x} - \frac{\sin x}{x^2}$$

$$= x^2\left[\frac{-x\sin x - \cos x}{x^2} - \frac{x^2\cos x - 2x\sin x}{x^4}\right]$$

$$+ 2x\left[\frac{\cos x}{x} - \frac{\sin x}{x^2}\right]$$

$$= -x\sin x \cancel{-\cos x} - \cancel{\cos x} + \cancel{\frac{2\sin x}{x}} + \cancel{2\cos x} - \cancel{\frac{2\sin x}{x}}$$

$$= -x\sin x = -x^2\frac{\sin x}{x}$$

$\Rightarrow$ we have just proved

$$x^2\left(\frac{\sin x}{x}\right)'' + 2x\left(\frac{\sin x}{x}\right)' + x^2\frac{\sin x}{x} = 0$$

$$\longrightarrow j_0(x) = \frac{\sin x}{x}.$$

$\lim_{x\to 0} j_0(x) = 1$ so $j_0(x)$ is regular at $x=0$.

(6) To find the eigenvalues, fit the boundary conditions

$$A_n(r) = j_0\left(\frac{\sqrt{\lambda}}{C_0}r\right) \Rightarrow A_n(R) = 0 \Rightarrow \frac{\sin\left(\frac{\sqrt{\lambda_n}}{C_0}R\right)}{\frac{\sqrt{\lambda_n}}{C_0}R} = 0$$

So $\sqrt{\lambda_n}\,\frac{R}{C_0} = n\pi$ or $\lambda_n = \frac{n^2\pi^2 C_0^2}{R^2}$

$\Rightarrow$ The fundamental is the lowest value (non-zero)

$$\lambda_{\text{fund.}} = \frac{\pi^2 C_0^2}{R^2} \rightarrow \text{quite close to } \frac{\sqrt{21}}{2}\frac{C_0^2}{R^2}.$$

(7) If $c(r) = C_s + C_m\left(1-\frac{r}{R}\right)$ then we can estimate the frequencies of the higher-order modes using

$$\lambda_n = \left(\frac{n\pi}{\int_0^R \sqrt{\frac{W(r')}{p(r')}}\,dr'}\right)^2 = \left(\frac{n\pi}{\int_0^R \sqrt{\frac{1}{c^2(r')}}\,dr'}\right)^2$$

$$= \left(\frac{n\pi}{\int_0^R \frac{dr'}{C_s + C_m\left(1-\frac{r'}{R}\right)}}\right)^2$$

the integral is

$$\int_0^R \frac{dr'}{C_s + C_m\left(1-\frac{r'}{R}\right)} = -\left[\frac{R}{C_m}\ln\left(C_s + C_m\left(1-\frac{r'}{R}\right)\right)\right]_0^R$$

$$= -\frac{R}{C_m}\left\{\ln(C_s) - \ln(C_s + C_m)\right\}$$

$$= \frac{R}{C_m}\ln\left(\frac{C_s + C_m}{C_s}\right)$$

So $$\lambda_n = n^2\pi^2\frac{C_m^2}{R^2}\cdot\frac{1}{\left[\ln\left(\frac{C_s+C_m}{C_s}\right)\right]^2}$$

$$\omega_n \simeq \sqrt{\lambda_n} = n\pi\frac{C_m}{R}\cdot\frac{1}{\ln\left(\frac{2.5\,10^7 + 10^5}{10^5}\right)} \simeq \frac{n\pi C_m}{R}\cdot\frac{1}{\ln(250)}$$

$$\simeq \frac{n\pi C_m}{5.5R}$$