FINAL 06 ANSWERS

Problem 1

① Eigenmodes of $c^2 u_{xx} = -\lambda^2 u$

$$V_n(x) = \begin{cases} \cos\left(\frac{\lambda_n}{c}x\right) \\ \sin\left(\frac{\lambda_n}{c}x\right) \end{cases}$$

$V_n(0) = 0 \Rightarrow$ no need of cos functions

$V_n(L) = 0 \Rightarrow \frac{\lambda_n}{c} = \frac{n\pi}{L}$

② So decompose solution on basis of spatial eigenfunctions

$$u(x,t) = \sum_n{}' f_n(t)\, V_n(x) = \sum_n{}' f_n(t) \sin\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow k_n = \frac{n\pi}{L}$$

③ Plugging this into $u_{tt} - c^2 u_{xx} = W(x,t)$

$$\Rightarrow \sum_n{}' \ddot{f}_n(t) \sin\left(\frac{n\pi x}{L}\right) + \frac{n^2\pi^2}{L^2}c^2 \sin\left(\frac{n\pi x}{L}\right) f_n(t) = W(x,t)$$

Using the orthogonality of the eigenfunctions

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \frac{L}{2}\delta_{mn}$$

$$\Rightarrow \ddot{f}_n(t) + \frac{n^2\pi^2 c^2}{L^2} f_n(t) = \frac{2}{L}\int_0^L W(x,t) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{L}\int_0^L x(L-x) \sin^2(\omega t) \sin\frac{n\pi x}{L}\, dx$$

$$= \frac{2}{L}\int_0^L x(L-x) \sin\left(\frac{n\pi x}{L}\right) dx \cdot \sin^2(\omega t)$$

$$= \alpha_n \sin^2(\omega t)$$

with $\alpha_n = \frac{2}{L}\int_0^L x(L-x)\sin\left(\frac{n\pi x}{L}\right)dx$

So $$= \frac{2}{L}\left\{ \cancelto{0}{\left[-x(L-x)\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right]_0^L} + \int_0^L (L-2x)\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)dx \right\}$$

$$= \frac{2}{L}\left\{ \cancelto{0}{\left[\left(\frac{L}{n\pi}\right)^2(L-2x)\sin\left(\frac{n\pi x}{L}\right)\right]_0^L} - \int_0^L \left(\frac{L}{n\pi}\right)^2(-2)\sin\left(\frac{n\pi x}{L}\right)dx \right.$$

$$= \frac{2}{L}\cdot 2\left(\frac{L}{n\pi}\right)^2\cdot\left[-\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right]_0^L$$

$$= \frac{4L^2}{n^3\pi^3}\left(1-\cos n\pi\right)$$

④ $u(x,0) = 0 \iff \sum_n f_n(0)\sin\left(\frac{n\pi x}{L}\right) = 0$

$\iff f_n(0) = 0$

$u_t(x,0) = 0 \iff \sum_n \dot{f}_n(0)\sin\left(\frac{n\pi x}{L}\right) = 0$

$\iff \dot{f}_n(0) = 0$

⑤ Solution of homogeneous equation for $f_n(t)$ is

$$A_n\cos\left(\frac{n\pi c}{L}t\right) + B_n\sin\left(\frac{n\pi c}{L}t\right)$$

Particular solution of forced problem:

Use $\sin^2(\omega t) = \frac{1}{2}(1-\cos 2\omega t)$

so try $P.I. = b_n + d_n\cos 2\omega t$

$\Rightarrow (\cos 2\omega t)\,d_n(-4\omega^2) + \left(+\frac{n^2\pi^2c^2}{L^2}\right)(b_n + d_n\cos 2\omega t) = \frac{\alpha_n}{2}(1-\cos 2\omega t)$

$$\Rightarrow \begin{cases} -4\omega^2 d_n + \frac{n^2\pi^2c^2}{L^2}d_n = -\frac{\alpha_n}{2} \\ +\frac{n^2\pi^2c^2}{L^2}b_n = \frac{\alpha_n}{2} \end{cases}$$

$$\Rightarrow \begin{cases} d_n = \dfrac{\alpha_n}{2\left[4\omega^2 - \dfrac{n^2\pi^2c^2}{L^2}\right]} \\[2ex] b_n = + \dfrac{\alpha_n L^2}{2n^2\pi^2c^2} \end{cases}$$

For $f_n(0) = 0$ we need $A_n + b_n + d_n = 0$

$\dot{f}_n(t) = 0$ we need $\frac{n\pi c}{L} B_n = 0$

So finally

$$\begin{cases} f_n(t) = -(b_n + d_n)\cos\left(\frac{n\pi c}{L}t\right) + b_n + d_n \cos(2\omega t) \\ u(x,t) = \sum_n' f_n(t) \sin\left(\frac{n\pi x}{L}\right) \end{cases}$$

$$\Rightarrow u(x,t) = \sum_n -\left(\frac{\alpha_n L^2}{2n^2\pi^2c^2} + \frac{\alpha_n}{2\left[4\omega^2 - \frac{n^2\pi^2c^2}{L^2}\right]}\right)\cos\left(\frac{n\pi c t}{L}\right)$$

$$+ \frac{\alpha_n}{2}\frac{L^2}{n^2\pi^2c^2} + \frac{\alpha_n}{2\left[4\omega^2 - \frac{n^2\pi^2c^2}{L^2}\right]}\cos 2\omega t$$

$$= \sum_n \frac{\alpha_n L^2}{2} \left\{ -\left(\frac{1}{n^2\pi^2c^2} + \frac{1}{4\omega^2L^2 - n^2\pi^2c^2}\right)\cos\left(\frac{n\pi c t}{L}\right) + \frac{1}{n^2\pi^2c^2} + \frac{1}{4\omega^2L^2 - n^2\pi^2c^2}\cos(2\omega t) \right\}$$

$$= \sum_n' \frac{\alpha_n L^2}{2n^2\pi^2c^2} \left\{ -\left(1 + \frac{1}{\frac{4\omega^2L^2}{n^2\pi^2c^2} - 1}\right)\cos\left(\frac{n\pi c t}{L}\right) + 1 + \frac{1}{\frac{4\omega^2L^2}{n^2\pi^2c^2} - 1}\cos 2\omega t \right\}$$

$$= \sum' \frac{4L^4}{2n^5\pi^5c^2} \left\{ -\left(1 + \frac{1}{\frac{4\omega^2L^2}{n^2\pi^2c^2} - 1}\right)\cos\left(\frac{n\pi c t}{L}\right) + 1 + \frac{1}{\frac{4\omega^2L^2}{n^2\pi^2c^2} - 1}\cos 2\omega t \right\}$$

Problem 2 FINAL 2006

① Plug $v_0(\theta)$ and $v_1(\theta)$ into equation:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d1}{d\theta}\right) = 0 = -\lambda_0 v_0 \Rightarrow \lambda_0 = 0$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\cos\theta}{d\theta}\right) = -\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin^2\theta\right)$$

$$= -\frac{2\sin\theta\cos\theta}{\sin\theta} = -2\cos\theta = -\lambda_1 v_1$$

$$\Rightarrow \lambda_1 = 2$$

② • The boundary conditions are independent of ϕ

→ we expect the solution to be independent of ϕ as well.

• The boundary conditions can be written as

$\sum_1 \alpha_n v_n(\theta)$ with $v_n(\theta) =$ some eigenfunctions of the θ- operator

⇒ we expect the solution to have the same property

$$\rightarrow T(r,\theta,\phi) = \tilde{T}(r,\theta) \leftarrow \text{no } \phi \text{ dependence}$$

$$= f_0(r)v_0(\theta) + f_1(r)v_1(\theta)$$

$$= f_0(r) + f_1(r)\cos\theta.$$

Plugging this into the equation ⇒

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df_0}{dr}\right) + \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df_1}{dr}\right)\cos\theta + \frac{1}{r^2}\frac{f_1(r)}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\cos\theta}{d\theta}\right) = 0$$

$$\Leftrightarrow \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df_0}{dr}\right) + \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df_1}{dr}\right)\cos\theta - \frac{2}{r^2}f_1(r)\cos\theta = 0$$

which implies
$$\begin{cases} \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df_0}{dr}\right) = 0 \\ \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df_1}{dr}\right) - \frac{2}{r^2}f_1 = 0 \end{cases}$$

$$r^2 \frac{df_0}{dr} = \text{constant} = A$$

$$\Rightarrow \frac{df_0}{dr} = \frac{A}{r^2} \Rightarrow f_0 = -\frac{A}{r} + B$$

Solving for f_1: try $f_1 = r^\alpha$: $\frac{d^2 f_1}{dr^2} + \frac{2}{r}\frac{df_1}{dr} - \frac{2}{r^2} f_1 = 0$

then $\alpha(\alpha-1) + 2\alpha - 2 = 0$

$$\Rightarrow \alpha(\alpha-1) + 2(\alpha-1) = 0$$

$$\Leftrightarrow (\alpha+2)(\alpha-1) = 0 \Rightarrow \begin{cases} \alpha = 1 \\ \alpha = -2 \end{cases}$$

So $f_1 = Cr + \frac{D}{r^2}$

So finally $T(r,\theta,\phi) = -\frac{A}{r} + B + \left(Cr + \frac{D}{r^2}\right)\cos\theta$.

③ Fitting to boundary conditions:

$$T(a,\theta,\phi) = T_0 \Leftrightarrow$$

$$-\frac{A}{a} + B + \left(Ca + \frac{D}{a^2}\right)\cos\theta = T_0$$

$$\Rightarrow \begin{cases} -\frac{A}{a} + B = T_0 & (*) \\ Ca + \frac{D}{a^2} = 0 & (**) \end{cases}$$

$$T(b,\theta,\phi) = T_1 + T_2\cos\theta \Leftrightarrow$$

$$-\frac{A}{b} + B + \left(Cb + \frac{D}{b^2}\right)\cos\theta = T_1 + T_2\cos\theta$$

$$\Rightarrow \begin{cases} -\frac{A}{b} + B = T_1 & (\dagger) \\ Cb + \frac{D}{b^2} = T_2 & (\dagger\dagger) \end{cases}$$

From $(**)$ $\Rightarrow$ $C = -\frac{D}{a^3}$

$(\dagger\dagger) \Rightarrow -\frac{Db}{a^3} + \frac{D}{b^2} = T_2 \Rightarrow D = \frac{T_2}{\frac{1}{b^2} - \frac{b}{a^3}}$

From $(*)-(\dagger)$: $-\frac{A}{a} + \frac{A}{b} = T_0 - T_1$

$$\Rightarrow A = \frac{T_0 - T_1}{} = \frac{ab(T_0 - T_1)}{}$$

and finally $B = T_0 + \frac{A}{a} = T_0 + \frac{b(T_0 - T_1)}{a-b}$

$$\Rightarrow \quad T(r,\theta,\phi) = -\frac{ab(T_0 - T_1)}{a-b}\frac{1}{r} + T_0 + \frac{b(T_0-T_1)}{a-b}$$
$$+ \left(-\frac{1}{a^3} r + \frac{1}{r^2}\right)\left(\frac{T_2}{\frac{1}{b^2} - \frac{b}{a^3}}\right)\cos\theta$$

Note: simplify with $D = \frac{T_2 b^2 a^3}{a^3 - b^3}$

$$C = -\frac{D}{a^3} = \frac{T_2 b^2}{b^3 - a^3}$$

$$\Rightarrow \quad T(r,\theta,\phi) = T_0 + \frac{b(T_1 - T_0)}{b-a}\left(1 - \frac{a}{r}\right)$$
$$+ \left(\frac{1}{r^2} - \frac{r}{a^3}\right)\frac{T_2 b^2 a^3}{a^3 - b^3}\cos\theta$$

$$= T_0 + \frac{b(T_1 - T_0)}{b-a}\left(1 - \frac{a}{r}\right)$$
$$+ \left(\frac{a^2}{r^2} - \frac{r}{a}\right)\frac{T_2 b^2 a}{a^3 - b^3}\cos\theta$$

<u>Check</u>: at $r = a$: $T = T_0$ ✓

at $r = b$: $T = T_0 + \frac{b(T_1 - T_0)}{b-a}\left(1 - \frac{a}{b}\right)$
$$+ \left(\frac{a^2}{b^2} - \frac{b}{a}\right)\frac{T_2 b^2 a}{a^3 - b^3}\cos\theta$$
$$= \cancel{T_0} + T_1 - \cancel{T_0} + T_2\cos\theta$$
$$= T_1 + T_2\cos\theta \quad ✓$$

Problem 3

$$\begin{cases} u_{tt} - 4u_{xx} = e^x + \sin t \\ u(x,0) = 0 \\ u_t(x,0) = \frac{1}{1+x^2} \end{cases}$$

$\Rightarrow$ Use d'Alembert's formula with $c = 2$

$F(x,t) = e^x + \sin t$

$f(x) = 0$

$g(x) = \frac{1}{1+x^2}$

$$\Rightarrow u(x,t) = 0 + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds + \frac{1}{2c}\iint_\Delta F(x',t')\,dx'dt'$$

First integral:

$$\int_{x-ct}^{x+ct} \frac{1}{1+s^2}\,ds = \operatorname{atan}(x+ct) - \operatorname{atan}(x-ct)$$

Second integral

$$\iint_\Delta F(\xi,\tau)\,d\xi d\tau = \int_{\tau=0}^{t}\int_{\xi = x-c(t-\tau)}^{x+c(t-\tau)} F(\xi,\tau)\,d\xi d\tau$$

$$= \int_0^t \int_{\xi = x-c(t-\tau)}^{\xi = x+c(t-\tau)} (e^\xi + \sin\tau)\,d\xi d\tau$$

$$= \int_0^t \left[e^{x+c(t-\tau)} - e^{x-c(t-\tau)}\right] d\tau$$

$$+ \int_0^t 2c(t-\tau)\sin\tau\,d\tau$$

$$= e^{x+ct}\int_0^t e^{-c\tau}\,d\tau - e^{x-ct}\int_0^t e^{c\tau}\,d\tau$$

$$+ 2ct\int_0^t \sin\tau\,d\tau - 2c\int_0^t \tau\sin\tau\,d\tau$$

$$= e^{x+ct}\left[-\frac{1}{c}e^{-c\tau}\right]_0^t - e^{x-ct}\left[\frac{1}{c}e^{c\tau}\right]_0^t$$

$$+ 2ct\left[-\cos\tau\right]_0^t - 2c\left[-\tau\cos\tau\right]_0^t - 2c\int_0^t \cos\tau\, d\tau$$

$$= e^{x+ct}\left[-\frac{1}{c}e^{-ct} + \frac{1}{c}\right] - e^{x-ct}\left[\frac{1}{c}e^{ct} - \frac{1}{c}\right]$$

$$+ 2ct\left[1-\cos t\right] + 2ct\cos t - 2c\sin t$$

$$= \cancel{-\frac{1}{c}e^{x}} + \frac{1}{c}e^{x+ct} \cancel{-\frac{1}{c}e^{x}} + \frac{1}{c}e^{x-ct} + 2ct - 2c\sin t$$

$$= \frac{2}{c}e^{x}\cosh(ct) + 2c(t-\sin t) - \frac{2}{c}e^{x}$$

$$\Rightarrow \quad u(x,t) = \frac{1}{2c}\left[\operatorname{atan}(x+ct) - \operatorname{atan}(x-ct)\right]$$
$$+ \frac{1}{c^2}e^{x}\cosh(ct) + t - \sin t - \frac{1}{c^2}e^{x}$$

Check: ~~$u_{tt} =$~~

$$u(x,0) = \frac{1}{2c}\left[\operatorname{atan} x - \operatorname{atan} x\right] + \frac{e^x}{c^2} - \frac{e^x}{c^2} = 0$$

$$u_t(x,t) = \frac{1}{2c}\left[\frac{c}{1+(x+ct)^2} + \frac{c}{1+(x-ct)^2}\right]$$
$$+ \frac{1}{c}e^{x}\sinh(ct) + 1 - \cos t$$

$$u_t(x,0) = \frac{1}{1+x^2} \quad 1+1-1 = \frac{1}{1+x^2}$$ as required.

$$\frac{d^2}{dx^2}\left(\operatorname{atan}(x+ct)\right) = \operatorname{atan}''(u)\big|_{u=x+ct}$$
$$\frac{d^2}{dt^2}\left(\operatorname{atan}(x+ct)\right) = c^2\operatorname{atan}''(u)\big|_{u=x+ct}$$
$$\Rightarrow \quad u_{tt} - c^2u_{xx} = \sin t + e^x$$ as required.

$$\begin{cases} u_{xx} + (1+y^2)^2 u_{yy} + 2y(1+y^2) u_y = 0 \\ u(x,0) = x \\ u_y(x,0) = -2x \end{cases}$$

- an elliptic equation
- Change of variables $(x,y) \to (\xi, \eta)$.

Define $\phi = \xi + i\eta$ then characteristics of ϕ

are found by integrating

$$\frac{dy}{dx} = \pm i\sqrt{(1+y^2)^2} = \pm i(1+y^2)$$

$$\Rightarrow \frac{dy}{1+y^2} = \pm i\,dx$$

$$\Rightarrow \operatorname{atan} y = \pm ix + \phi$$

$$\Rightarrow \begin{cases} \xi = \operatorname{atan} y \\ \eta = x \end{cases}$$

$$\Rightarrow \begin{cases} \xi_x = 0 & \xi_y = \frac{1}{1+y^2} & \xi_{yy} = -\frac{2y}{(1+y^2)^2} \\ \eta_x = 1 & \eta_y = 0 & \Rightarrow u_y = \frac{1}{1+y^2} u_\xi \text{ and} \end{cases}$$

$$u_{xx} = \cancel{\xi_x^2 u_{\xi\xi}} + 2\cancel{\xi_x \eta_x u_{\xi\eta}} + \eta_x^2 u_{\eta\eta} + \cancel{u_\xi \xi_{xx}} + \cancel{u_\eta \eta_{xx}}$$

$$= u_{\eta\eta}$$

$$u_{yy} = \xi_y^2 u_{\xi\xi} + \cancel{2\xi_y \eta_y u_{\xi\eta}} + \cancel{\eta_y^2 u_{\eta\eta}} + u_\xi \xi_{yy} + \cancel{u_\eta \eta_{yy}}$$

$$= \frac{1}{(1+y^2)^2} u_{\xi\xi} - \frac{2y}{(1+y^2)^2} u_\xi$$

$$\Rightarrow u_{xx} + (1+y^2)^2 u_{yy} + 2y(1+y^2) u_y = u_{\xi\xi} + u_{\eta\eta} - \cancel{\frac{2y}{(1+y^2)} u_\xi} + \cancel{2y u_\xi} = 0$$

$\Rightarrow \quad u_{\xi\xi} + u_{\eta\eta} = 0$

Try $\quad u = (a\xi + b)(c\eta + d)$

$\Rightarrow \quad u(x,y) = \left[a \cdot \operatorname{atan} y + b\right]\left[cx + d\right]$

$u(x,0) = b\left[cx + d\right]$

$u_y(x,y) = \frac{a}{1+y^2}\left[cx + d\right]$

So $\quad u_y(x,0) = a\left[cx + d\right]$

$\Rightarrow \quad b\left[cx+d\right] = x \quad \Rightarrow \quad bc = 1, \quad d = 0$

$a\left[cx+d\right] = -2x \quad \Rightarrow \quad ac = -2, \quad d = 0$

$\Rightarrow \quad u(x,y) = c\left[a \operatorname{atan} y + b\right]x$

$= \left[-2\operatorname{atan} y + 1\right]x$

Check: $\quad u(x,0) = x$

$u_y(x,y) = \frac{-2x}{1+y^2}$

So $\quad u_y(x,0) = -2x$

$u_{xx} = 0$

$u_{yy} = \frac{-2x(-2y)}{(1+y^2)^2} = \frac{4xy}{(1+y^2)^2}$

So $\quad u_{xx} + (1+y^2)^2 u_{yy} + 2y(1+y^2)u_y$

$= 0 + 4xy + 2y(-2x) = 0$ indeed.