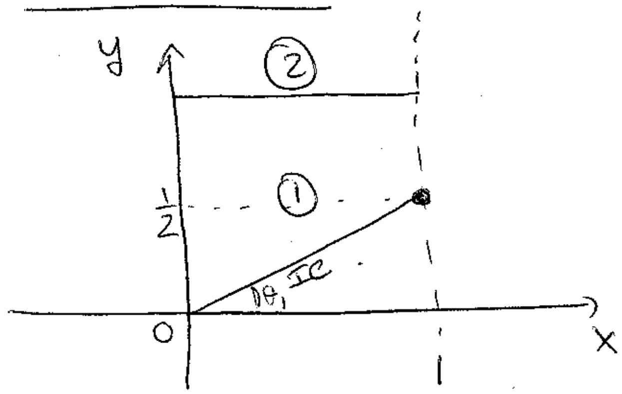
FINAL Q1



$$\begin{cases} u_x^2 + u_y^2 = n_0^2 & \text{in } 1 \\ u_x^2 + u_y^2 = n_1^2 & \text{in } 2. \end{cases}$$

$$\text{CEs:} \begin{cases} x_\tau = 2p \\ y_\tau = 2q \\ u_\tau = 2n^2 \\ p_\tau = 0 \\ q_\tau = 0 \end{cases}$$

in 1: IC is $y = \frac{1}{2}x \rightarrow u(x, \frac{1}{2}x) = 1$

$$x_0 = s$$
$$y_0 = \frac{1}{2}s$$
$$u_0 = 1$$

so $\left.\begin{array}{l} p_0^2 + q_0^2 = n_0^2 \\ p_0 + \frac{1}{2}q_0 = 0 \end{array}\right\} \Rightarrow \begin{cases} \frac{1}{4}q_0^2 + q_0^2 = n_0^2 \\ p_0 = -\frac{1}{2}q_0 \end{cases}$

$$\Rightarrow \frac{5}{4}q_0^2 = n_0^2 \Rightarrow q_0 = \frac{2n_0}{\sqrt{5}} \quad p_0 = -\frac{n_0}{\sqrt{5}}$$

$$\Rightarrow \quad p(s) = p_0 = -\frac{n_0}{\sqrt{5}} \qquad u(s) = 2n^2\tau + u_0 = 2n^2\tau + 1$$
$$q(s) = q_0 = \frac{2n_0}{\sqrt{5}} \qquad x(s) = -\frac{2n_0}{\sqrt{5}}\tau + s$$
$$y(s) = \frac{4n_0}{\sqrt{5}}\tau + \frac{1}{2}s$$

$$\Rightarrow \quad x - 2y = -\frac{2n_0}{\sqrt{5}}\tau - \frac{8n_0}{\sqrt{5}}\tau = -\frac{10n_0}{\sqrt{5}}\tau$$

so $\tau = -\frac{\sqrt{5}}{10n_0}(x-2y)$

$$\Rightarrow u(x,y) = -2n_0^{\cancel{2}} \cdot \frac{\sqrt{5}}{10\cancel{n_0}}(x-2y) + 1 = -\frac{n_0}{\sqrt{5}}(x-2y) +$$

$\Rightarrow$ on $y = 1$, $u(x,1) = 1 - \frac{n_0}{\sqrt{5}}(x-2)$

In ② : IC is $u(x,1) = 1 - \frac{n_0}{\sqrt{5}}(x-2)$

$$x_0 = s$$
$$y_0 = 1$$
$$u_0 = 1 - \frac{n_0}{\sqrt{5}}(s-2)$$

$$\left.\begin{array}{l} p_0^2 + q_0^2 = n_1^2 \\ p_0 = -\frac{n_0}{\sqrt{5}} \end{array}\right\} q_0^2 = n_1^2 - \frac{n_0^2}{5}$$

so
$$p(s) = p_0 = -\frac{n_0}{\sqrt{5}}$$
$$q(s) = q_0 = \sqrt{n_1^2 - \frac{n_0^2}{5}}$$

note: q_0 exists $\Rightarrow n_1^2 > \frac{n_0^2}{5}$

then
$$x(s) = -\frac{2n_0}{\sqrt{5}}\tau + s$$
$$y(s) = 2\sqrt{n_1^2 - \frac{n_0^2}{5}}\,\tau + 1$$
$$u(s) = 2n_1^2\tau + 1 - \frac{n_0}{\sqrt{5}}(s-2)$$

thus time $\tau = \frac{y-1}{2\sqrt{n_1^2 - \frac{n_0^2}{5}}}$

so
$$s = x + \frac{\cancel{2}n_0}{\sqrt{5}}\,\frac{y-1}{\cancel{2}\sqrt{n_1^2 - \frac{n_0^2}{5}}}$$
$$= x + \frac{y-1}{\sqrt{5\frac{n_1^2}{n_0^2} - 1}}$$

So
$$u(s) = \cancel{2}n_1^2\,\frac{y-1}{\cancel{2}\sqrt{n_1^2 - \frac{n_0^2}{5}}} + 1 - \frac{n_0}{\sqrt{5}}\left[x - 2 + \frac{y-1}{\sqrt{5\frac{n_1^2}{n_0^2} - 1}}\right]$$
$$= \frac{n_1(y-1)}{\sqrt{1 - \frac{n_0^2}{5n_1^2}}} + 1 - \frac{n_0}{\sqrt{5}}(x-2)$$
$$- \frac{n_0}{\sqrt{5}}\,\frac{y-1}{\sqrt{5\frac{n_1^2}{n_0^2} - 1}}$$

$$= 1 - \frac{n_0}{\sqrt{5}}(x-2) + (y-1)\left[\frac{n_1}{\sqrt{1-\frac{n_0^2}{5n_1^2}}} - \frac{n_0}{\sqrt{5}}\frac{1}{\sqrt{5\frac{n_1^2}{n_0^2}-1}}\right].$$

$$= 1 - \frac{n_0}{\sqrt{5}}(x-2) + (y-1)\left[\frac{\sqrt{5}\,n_1^2}{\sqrt{5n_1^2-n_0^2}} - \frac{n_0^2}{\sqrt{5}}\frac{1}{\sqrt{5n_1^2-n_0^2}}\right]$$

$$= 1 - \frac{n_0}{\sqrt{5}}(x-2) + \frac{(y-1)}{\sqrt{5}}\left(\frac{5n_1^2-n_0^2}{\sqrt{5n_1^2-n_0^2}}\right)$$

$$= 1 - \frac{n_0}{\sqrt{5}}(x-2) + \frac{y-1}{\sqrt{5}}\sqrt{5n_1^2-n_0^2}.$$

So lines of constant u satisfy

$$-\frac{n_0}{\sqrt{5}}x + \frac{y}{\sqrt{5}}\sqrt{5n_1^2-n_0^2} = \text{constant}$$

$$\Rightarrow \quad y = x\frac{n_0}{\sqrt{5n_1^2-n_0^2}} + \text{constant}.$$

<u>Recap</u>: in ①, lines of constant u satisfy

$$y = \frac{1}{2}x + \text{const}.$$

in ②

$$y = \frac{n_0}{\sqrt{5n_1^2-n_0^2}}x + \text{const}$$

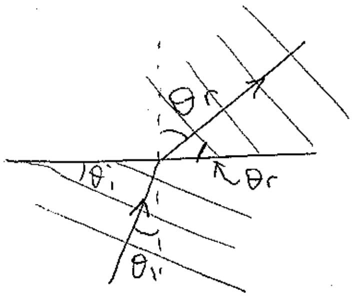
The angle with the horizontal, in ①, is

$$\tan\theta_1 = \frac{1}{2}$$

in ②: $\tan\theta_2 = \dfrac{n_0}{\sqrt{5n_1^2-n_0^2}}$

Snell's Law

$$n_i \sin\theta_i = n_r \sin\theta_r$$



→ same for the angles of wave fronts with interface.

$$\sin\theta_r = \frac{n_i}{n_r}\sin\theta_i$$

→ if that # > 1 then problem

So we need $\sin\theta_1 = \dfrac{\frac{1}{2}}{\sqrt{(\frac{1}{2})^2+1}} = \dfrac{1}{\sqrt{5}}$

$$\sin\theta_2 = \frac{\dfrac{n_0}{\sqrt{5n_1^2-n_0^2}}}{\sqrt{\dfrac{n_0^2}{5n_1^2-n_0^2}+1}} = \frac{n_0}{\sqrt{5}\,n_1}$$

- the condition for refraction is $\frac{n_0}{n_1}\frac{1}{\sqrt{5}} \leq 1$
  → as found earlier

- Snells law is indeed satisfied:

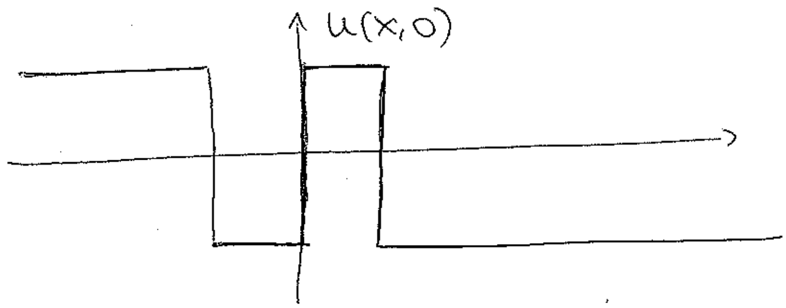
$$\sin\theta_2 = \frac{n_0}{n_1}\sin\theta_1.$$

Final Q2

Consider the conservation law

$$u_t + u u_x = 0$$

$$u(x,0) = \begin{cases} 1 & x < -1 \\ -1 & x \geq -1 \quad x < 0 \\ 1 & x \geq 0 \quad x < 1 \\ -1 & x \geq 1 \end{cases}$$



$F'(u) = u$

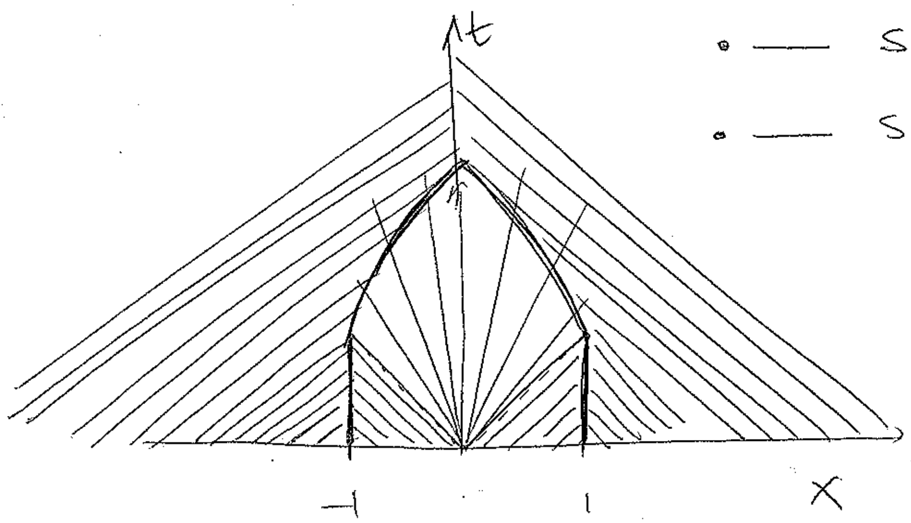
$F(u) = \frac{u^2}{2}$

$\phi(x) = u(x,0)$

① Characteristics are s. t

- for $s < -1$ then $t = \dfrac{x-s}{F'(\phi(s))} = \dfrac{x-s}{\phi(s)} = \dfrac{x-s}{1}$
  $= x - s$

similarly
- For $s \in [-1, 0)$ $\quad t = \dfrac{x-s}{-1} = s - x$
- —— $s \in [0, 1)$ $\quad t = x - s$
- —— $s \in [1, +\infty)$ $\quad t = s - x$.



② Rarefaction region

$$x = tF'(u) + s \quad \text{emanating from } s=0$$

$$\rightarrow \quad F'(u) = \frac{x}{t} = u$$

$$\rightarrow \quad u = \frac{x}{t} \text{ in that region.}$$

③ Shocks for $t \leq 1$

at $x=-1$ and $x=1$

$$\frac{d\gamma}{dt} = \frac{F(u_+) - F(u_-)}{u_+ - u_-} = \frac{\frac{1}{2}u_+^2 - \frac{1}{2}u_-^2}{u_+ - u_-}$$

$$= 0 \qquad \text{since } u_+ = -1, \; u_- = 1.$$

④ Shocks for $t \geq t_1, t \leq t_2$

Shock is between solution $u_- = 1$ and rarefaction solution $u_+ = \frac{x}{t}$

$$\text{So} \quad \frac{d\gamma}{dt} = \frac{\frac{1}{2}\frac{\cancel{x}^2}{t^2} - \frac{1}{2}}{\frac{\cancel{x}}{t} - 1} = \frac{1}{2}\frac{\frac{\cancel{x}^2}{t^2} - 1}{\frac{\cancel{x}}{t} - 1} = \frac{1}{2}\left(\frac{\cancel{x}}{t} + 1\right)$$

$$\text{So} \quad \frac{d\gamma}{dt} - \frac{1}{2}\frac{\gamma}{t} = \frac{1}{2}$$

IF method $\Rightarrow \mu = e^{-\frac{1}{2}\int\frac{1}{t}\,dt} = e^{-\frac{1}{2}\ln t} = t^{-\frac{1}{2}}$

$$\text{so} \quad \frac{d}{dt}\left(\gamma t^{-\frac{1}{2}}\right) = \frac{1}{2}t^{-\frac{1}{2}} = \frac{1}{2\sqrt{t}}$$

$$\gamma t^{-\frac{1}{2}} - \gamma(1) = \sqrt{t} - 1$$

$$\gamma = \gamma(1)t^{1/2} + t - \sqrt{t}$$

$$= (\gamma(1) - 1)\sqrt{t} + t.$$

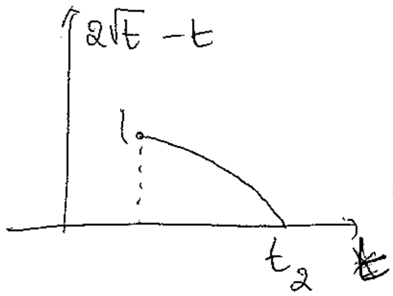
For the shock considered, $\gamma(1) = -1$

so $\quad \gamma = -2\sqrt{t} + t$.

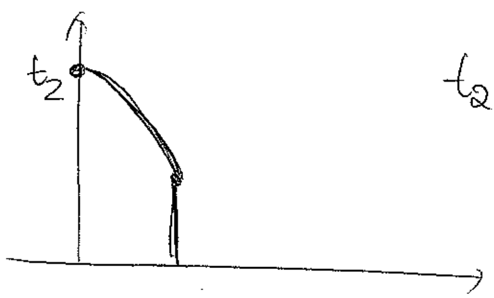
by symmetry or otherwise, we can see that

$\gamma = 2\sqrt{t} - t$ for the other shock.

$\Rightarrow$



$\Rightarrow$



t_2 is such that

$2\sqrt{t_2} - t_2 = 0$

$\Rightarrow 2 - \sqrt{t_2} = 0$

$\boxed{t_2 = 4}$.

Final Q3

$$u_{xx} - 2u_{xy} + 4e^{y} = 0$$

$$u(0,y) = f(y) \qquad u_x(0,y) = g(y)$$

① $\delta = 1 \rightarrow$ hyperbolic

② $\xi_x + (-1 \pm \sqrt{1})\,\xi_y = 0$

$$\Rightarrow \begin{cases} \xi_x = 0 \\ \eta_x - 2\eta_y = 0 \end{cases}$$

So ξ is independent of $x \rightarrow$ choose $\xi = y$

η: $\frac{dy}{dx} = -2 \Rightarrow y = -2x + \eta$

$$\Rightarrow \begin{cases} \xi = y & \xi_x = 0 \quad \xi_y = 1 \\ \eta = y + 2x & \eta_x = 2 \quad \eta_y = 1 \end{cases}$$

$$u_{xx} = 4u_{\eta\eta}$$

$$u_{xy} = 2u_{\xi\eta} + 2u_{\eta\eta}$$

$$\rightarrow 4u_{\eta\eta} - 2(2u_{\xi\eta} + 2u_{\eta\eta}) + 4e^{\xi} = 0$$

$$\rightarrow \boxed{u_{\xi\eta} = e^{\xi}}$$

$$u_\eta = e^{\xi} + \mathcal{F}(\eta)$$

$$u = e^{\xi}\eta + F(\eta) + G(\xi).$$

$$u = e^{y}(y+2x) + F(y+2x) + G(y)$$

at $x=0$

$$u = ye^{y} + F(y) + G(y) = f(y)$$

$$u_x = 2e^{y} + 2F'(y) = g(y)$$

$$\Rightarrow \quad F'(y) = \frac{1}{2}g(y) - e^{y}$$

so

$$F(y) = \frac{1}{2}\int_0^y g(s)\,ds + F(0) - e^{y}$$

$$ye^{y} + \frac{1}{2}\int_0^y g(s)\,ds + F(0) - e^{y} + G(y) = f(y)$$

so

$$G(y) = f(y) + (1-y)e^{y} - \frac{1}{2}\int_0^y g(s)\,ds - F(0)$$

$$\Rightarrow$$

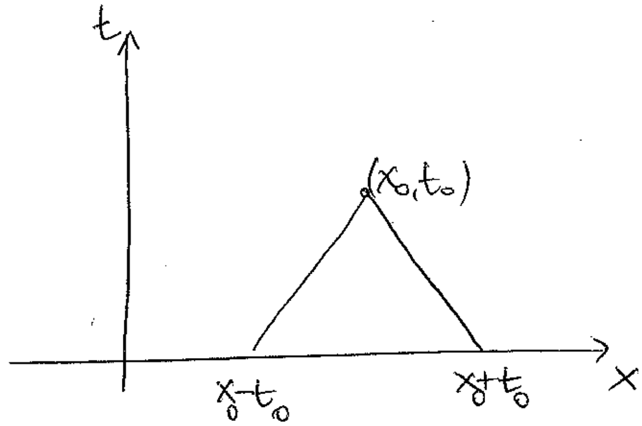
$$u = e^{y}(y+2x) + \frac{1}{2}\int_0^{y+2x} g(s)\,ds + \cancel{F(0)} - e^{y+2x}$$

$$+ f(y) + (1-y)e^{y} - \frac{1}{2}\int_0^y g(s)\,ds - \cancel{F(0)}$$

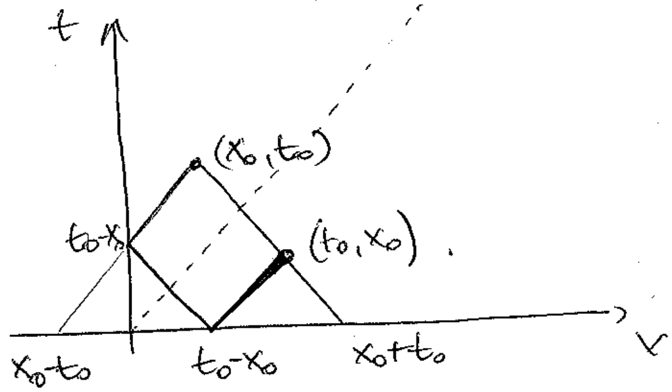
$$= e^{y}(2x+1) - e^{y+2x} + f(y) + \frac{1}{2}\int_y^{y+2x} g(s)\,ds.$$

FINAL Q4

For $x_0 > t_0$ then
no problem



For $x_0 < t_0$ then



$$u(x_0,t_0) + u(t_0-x_0,0) = u(t_0-x_0,0) + u(t_0,x_0)$$

$$u(x_0,t_0) = h(t_0-x_0) - f(t_0-x_0) + u(t_0,x_0).$$

$$= h(t_0-x_0) - f(t_0-x_0) + \frac{1}{2}\left(f(t_0-x_0) + f(t_0+x_0)\right) + \frac{1}{2}\int_{t_0-x_0}^{t_0+x_0} g(s)\,ds.$$

$$= h(t_0-x_0) + \frac{1}{2}\left[f(t_0+x_0) - f(t_0-x_0)\right] + \frac{1}{2}\int_{t_0-x_0}^{t_0+x_0} g(s)\,ds$$

FINAL Q4

see p 94

To solve problems of the kind

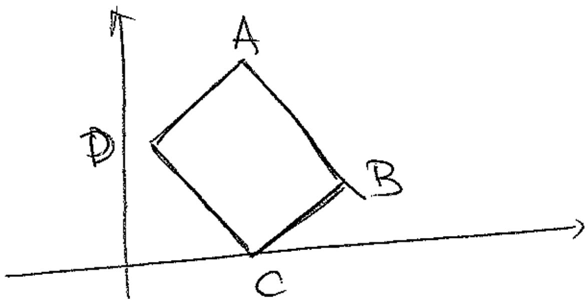
$$u_{tt} - u_{xx} = 0$$

$$u(0,t) = h(t) \qquad x > 0$$

$$u(x,0) = f(x) \qquad t > 0.$$

$$u_t(x,0) = g(x)$$

One uses the parallelogram identity



$u(A) + u(C) = u(B) + u(D)$
where the sides of the // are aligned with the characteristics.

1. Write d'Alemberts soln when $x > t$.
2. Draw graphically why d'A. soln doesn't work when $x < t$.
3. By considering a suitable parallelogram, ~~show that~~ use // identity to show that if $x < t$,

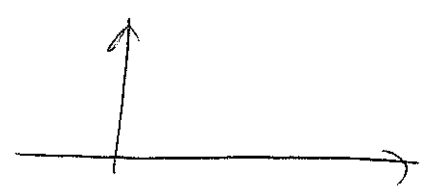
   $u(x,t) =$ __________

4. Solve the problem for

$$u_{tt} - u_{xx} = 0$$

$$u(0,t) = \frac{t}{1+t}$$

$$u(x,0) = u_t(x,0) = 0$$



$$u_{tt} + a^2 u_{xxxx} = 0 \qquad 0 < x < L$$

$\boxed{\text{FINAL } 5}$

$$u(x,0) = \_\_\_\_$$

u_t

$$X(x) = A\cosh\mu x + B\sinh\mu x + C\cos\mu x + D\sin\mu x$$

$$X(0) = 0$$
$$X'(0) = 0$$
$$X(L) = 0$$
$$X''(L) = 0.$$

$$\to \begin{cases} A + C = 0 \\ \mu B + \mu D = 0 \end{cases}$$

$$X(x) = A(\cosh\mu x - \cos\mu x) + B(\sinh\mu x - \sin\mu x).$$

$$A(\cosh\mu L - \cos\mu L) + B(\sinh\mu L - \sin\mu L) = 0$$

$$A \cancel{\mu^2}(\cosh\mu x + \cos\mu L) + \cancel{\mu^2} B(\sinh\mu L + \sin\mu L) = 0$$

~~$A(1+\mu^2)\cos\mu$~~

$$2A\cosh\mu L + 2B\sinh\mu L = 0$$

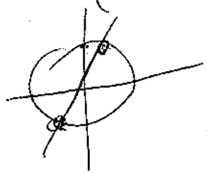
$$A\cos\mu L + B\sinh\mu L = 0.$$

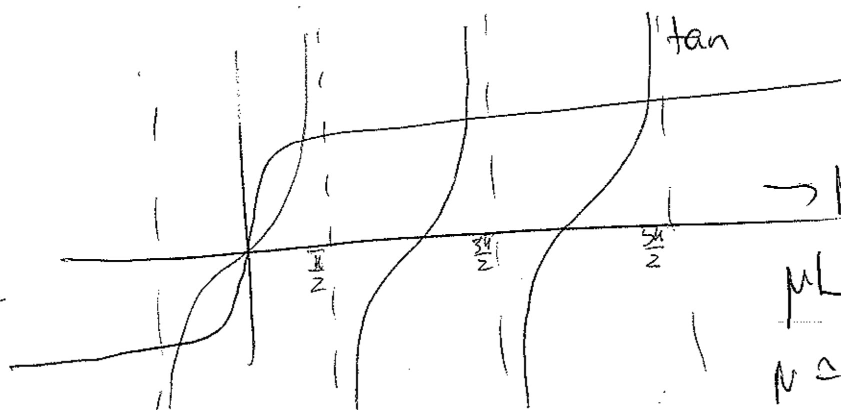
$\tanh = 1$

$\Rightarrow \tan\mu L \simeq 1$

$\Rightarrow \mu L \simeq \frac{\pi}{4} + \frac{2n\pi}{2}$

$= \left(\frac{4n+1}{4}\right)\pi$





$\to \mu L$ has ∞ sol^n.

$\mu L \simeq \cancel{\frac{\pi}{8}} + 2n\pi$

$\mu \simeq \cancel{\frac{\pi}{2L}} + \frac{2n\pi}{L}.$