Special case

Let's take the simplest possible example, that of constant sound-speed $c_s(r)$. Then

$$\frac{d}{dr}\left(r^2 \frac{dA}{dr}\right) - \ell(\ell+1)A = -\frac{\omega^2}{c_s^2} r^2 A$$

This is actually the equation for a spherical Bessel function. Indeed, spherical Bessel functions satisfy

$$\frac{d}{dx}\left(x^2 \frac{dA}{dx}\right) - \ell(\ell+1)A + x^2 A = 0. \Rightarrow \text{as above if } x = \frac{\omega}{c} r$$

$$A_\ell(x) = \begin{Bmatrix} j_\ell(x) \\ y_\ell(x) \end{Bmatrix}.$$

The $y_\ell(x)$ functions are singular at $x=0$ → discard.

At the surface, $A_\ell(x) = 0 \Rightarrow A_\ell\left(\frac{\omega}{c} R_*\right) = 0$.

⇒ this means that $\frac{\omega}{c} R_*$ are zeros of the Bessel $j_\ell(x)$ function. There is an infinite number of them, noted $z_{\ell n}$ (the n-th zero of the $j_\ell(x)$ function)

$$\Rightarrow \omega_{n\ell} = \frac{z_{\ell n}}{R_*} c_s.$$

⇒ Finally, putting everything together we have

$$p(r,\theta,\phi,t) = \sum_{n,\ell,m} A_{\ell n}(r) B_\ell^m(\theta) C_m(\phi) D_{n\ell m}(t) \quad \text{where}$$

$$A_{\ell n}(r) = j_\ell\left(z_{\ell n} \frac{r}{R_*}\right)$$

$$C_m(\phi) B_\ell^m(\theta) = Y_\ell^m(\theta,\phi) \leftarrow \text{spherical harmonic}$$

$$D_{n\ell m}(t) = a_{n\ell m} \cos(\omega_{\ell n} t) + b_{n\ell m} \sin(\omega_{\ell n} t)$$

where $\omega_{\ell n} = \frac{z_{\ell n}}{R_*} c_s$

General case ($C_s^2(r)$ not known).

S.L theory can help us find out more about the system

There are an infinity of eigenmodes $A(r)$ and associated eigenvalues ω, which we can reference by the index n ($n \in \mathbb{N}$)

→ the modes are orthogonal, and form a basis for all functions in $(0, R)$ satisfying
$|A(0)| < +\infty, \quad A(R) = 0$.

→ Note how the equation depends on l but not on m: the frequencies of the modes ω depend only on n and l but not on m

The lowest frequency of oscillation ($n=0$, for a given l) is such that

$$\omega^2 = \inf_{u \in V} \mathcal{R}(u) = \inf \frac{\int_0^R \left[p(r)\left(\frac{du}{dr}\right)^2 - q(r)\,u^2(r)\right]dr}{\int_0^R w(r)\,u^2(r)\,dr}$$

ensemble of all functions satisfying bcs

$$= \inf \frac{\int_0^R \left[r^2\left(\frac{du}{dr}\right)^2 + l(l+1)\,u^2(r)\right]dr}{\int_0^R \frac{r^2}{C_s^2(r)}\,u^2(r)\,dr}$$

- For the sake of simplicity, assume $C_s^2(r) = \dfrac{C_0^2}{1+\alpha\left(\frac{r}{R}\right)^2}$
  (which is not too far from actual profile $C_0 = 4\ 10^7$ cm/s, $\alpha \simeq 12$.)

- To ensure regularity @ origin, need $u(r) \propto r^l$

  To ensure $u=0$ @ $r=R$, need $u \propto (r-R)$

  → let $u(r) = r^l(R-r)$

$$\Rightarrow \frac{du}{dr} = l r^{l-1}(R_\odot - r) - r^l = r^{l-1}\left\{ l R_\odot - (l+1) r \right\}$$

So

$$\omega^2 \lesssim \frac{\int_0^{R_\odot} r^{2l}\left(lR_\odot-(l+1)r\right)^2 + l(l+1) r^{2l}(R_\odot-r)^2 \, dr}{\int_0^{R_\odot} \frac{r^2}{C_0^2} r^{2l}(R_\odot-r)^2\left(1+\alpha\left(\frac{r}{R}\right)^2\right) dr}$$

$$= \frac{C_0^2 R_\odot^{2l+3} \int_0^1 x^{2l}\left(l-(l+1)x\right)^2 + l(l+1)x^{2l}(1-x)^2 \, dx}{R_\odot^{2l+5}\int_0^1 x^{2l+2}(1-x)^2(1+\alpha x^2)\, dx}$$

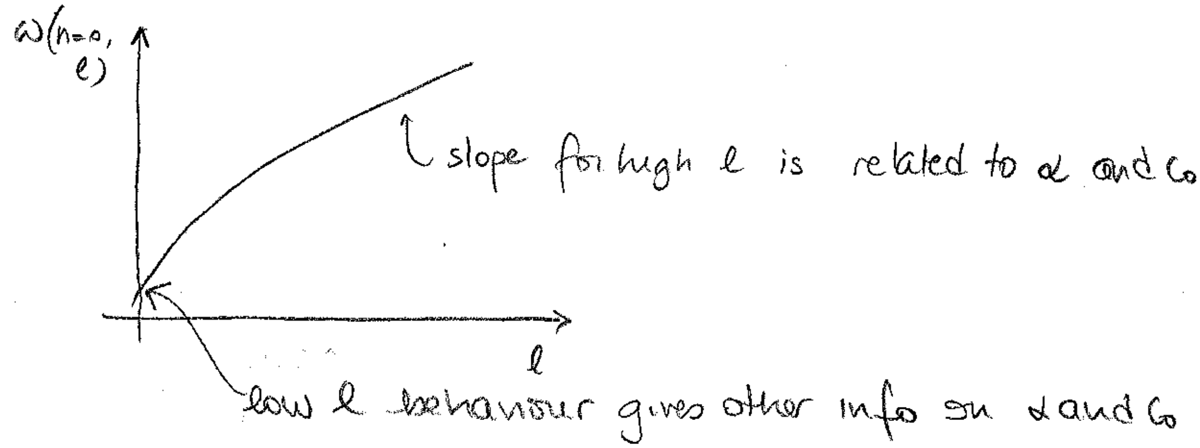
$$= \frac{C_0^2}{R_\odot^2} \frac{\int_0^1 x^{2l}\left[l(2l+1) - 4l(l+1)x + (l+1)(2l+1)x^2\right] dx}{\int_0^1 x^{2l+2}\left(1-2x+(1+\alpha)x^2 - 2\alpha x^3 + \alpha x^4\right) dx}$$

$$= \frac{C_0^2}{R_\odot^2} \frac{\left[\frac{l\cancel{(2l+1)}}{\cancel{(2l+1)}} - \frac{4l(l+1)}{2l+2} + \frac{(l+1)(2l+1)}{2l+3}\right]}{\left[\frac{1}{2l+3} - \frac{2}{2l+4} + \frac{1+\alpha}{2l+5} - \frac{2\alpha}{2l+6} + \frac{\alpha}{2l+7}\right]}$$

$$= \frac{C_0^2}{R_\odot^2} \frac{\frac{1}{2l+3}}{\frac{2}{(2l+3)(2l+4)(2l+5)} + \frac{2\alpha}{(2l+5)(2l+6)(2l+7)}}$$

As $l \to \infty$ then $\omega \simeq \frac{C_0}{R_\odot}\frac{\sqrt{2}\, l}{1+\alpha}$ as $l=0$ $\omega \to \frac{C_0}{R_\odot}\frac{1}{\frac{1}{10}+\frac{\alpha}{35}}$

note: oscillation period is $T(l) = \frac{2\pi}{\omega}$



$\Rightarrow$ the shape of $\omega(n=0, l)$ depends on C_0 and α in a way that can be fitted to data to <u>infer</u> C_0 and α $\Rightarrow$ HELIOSEISMOLOGY

(I) Introduction

We now consider forced linear PDES of the kind

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = F(x,t) \qquad \text{(Forced Wave Equation)}$$

$$\frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} = F(x,t) \qquad \text{(Forced heat Equation)}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = F(x,y) \qquad \text{("Forced Laplace equation" = Poisson equation)}$$

The method for solving these problems will be first illustrated through examples, then generalized in (II)

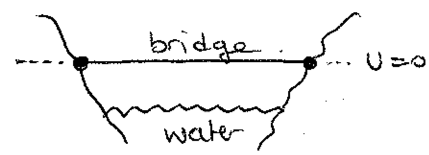
(1) Forced wave equation

Example : A bridge, suspended, and the wind forcing (cf Tacoma Narrows)

(in 2D : a metal plate, with some sand on it, and a speakerphone nearby; see Exploratorium).

let

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = F(x,t) & \\ u(x, t=0) = 0 & u(0,t) = 0 \\ u_t(x, t=0) = 0 & u(L,t) = 0 \end{cases}$$



→ A forced string, pinned at the sides, initially at rest.

Method: 1. Find the spatial eigenmodes $A_n(x)$ homogeneous problem - with same bcs

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 .$$

These will generally be mixtures of sines and cosines.

Here (see previous lectures)

$$A_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

2. Assume that the full solution can be written as

$$u(x,t) = \sum_{n=0}^{\infty} A_n(x) B_n(t)$$

and plug into PDE

$$\Rightarrow \sum_{n=0}^{\infty} A_n(x) \frac{d^2 B_n}{dt^2} - c^2 B_n(t) \frac{d^2 A_n}{dx^2} = F(x,t)$$

$$\Rightarrow \sum_{n=0}^{\infty} A_n(x) \frac{d^2 B_n}{dt^2} + \frac{c^2 n^2 \pi^2}{L^2} B_n(t) A_n(x) = F(x,t). \quad (*)$$

3. Note that $\int_0^L A_n(x) A_m(x)\, dx = 0 \quad \forall n \neq m$

$$= \frac{L}{2} \text{ if } n = m$$

<u>Aside</u>: The orthogonality property of the eigenfunctions will be true for a wide class of problems, see next chapter.

So take $(*)$ and multiply by $A_m(x)$, then integrate over $[0, L]$

$$\Rightarrow \frac{L}{2} \frac{d^2 B_m}{dt^2} + \frac{c^2 m^2 \pi^2}{L^2} \cdot \frac{L}{2} B_m = \int_0^L F(x,t) \sin\left(\frac{m\pi x}{L}\right) dx.$$

$$\Rightarrow \ddot{B}_m + \frac{c^2 m^2 \pi^2}{L^2} B_m = \frac{2}{L} \int_0^L F(x,t) \sin\left(\frac{m\pi x}{L}\right) dx$$

$$= f_m(t).$$

$\Rightarrow$ we now get a set of independent ODEs, one for each value of m. These are forced, second-order linear ODEs. (which you should be able to solve...)

Simple example

Suppose $F(x,t) = \sin\left(\frac{2\pi x}{L}\right)\cos(\omega t)$

then $f_m(t) = \int_0^L \frac{2}{L} \sin\left(\frac{2\pi x}{L}\right)\cos(\omega t)\sin\left(\frac{m\pi x}{L}\right) dx$

$$= \begin{cases} \frac{2}{L} \cdot \frac{L}{2} \cdot \cos\pi t & \text{if } m=2 \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow$ we have 2 types of ODES to solve:

$$\ddot{B}_2 + \frac{4c^2\pi^2}{L^2} B_2 = \cos\omega t$$

and $$\ddot{B}_m + \frac{c^2 m^2\pi^2}{L^2} B_m = 0 \qquad \forall m \neq 2.$$

For all of them, the solution to the homogeneous equation is

$$B_m(t) = \alpha_m \cos\left(\frac{cm\pi}{L} t\right) + \beta_m \sin\left(\frac{m\pi c t}{L}\right)$$

For the $B_2(t)$ function, we have to add a particular solution to the forced problem: here try

$$B_2^{PS}(t) = K \cos\omega t$$

a constant to be determined by plugging into equation

$$\Rightarrow \quad -K\omega^2 + \frac{4c^2\pi^2}{L^2} K = 1$$

$$\Rightarrow \quad K = \frac{1}{\frac{4c^2\pi^2}{L^2} - \omega^2}$$

$$\Rightarrow \quad B_2(t) = \alpha_2 \cos\left(\frac{2c\pi t}{L}\right) + \beta_2 \sin\left(\frac{2c\pi t}{L}\right) + \frac{\cos(\omega t)}{\frac{4c^2\pi^2}{L^2} - \omega}$$

$\Rightarrow$ So finally, we have

$$u(x,t) = \sum_{n=1}^{\infty} A_n(x)\, B_n(t) \qquad \text{where } A_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

$$B_n(t) = \alpha_n \cos\left(\frac{n\pi c t}{L}\right) + \beta_n \sin\left(\frac{n\pi c t}{L}\right) + \frac{\cos \pi t}{\frac{4c^2\pi^2}{L^2} - \omega^2}\, \delta_{n,2}$$

To find the arbitrary constants α_n, β_n, we fit the initial conditions:

$$u(x,0) = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} A_n(x) B_n(0) = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} A_n(x) \left[\alpha_n + \frac{1}{\frac{4c^2\pi^2}{L^2} - \omega^2}\, \delta_{n,2} \right] = 0$$

$$\Rightarrow \alpha_n + \frac{1}{\frac{4c^2\pi^2}{L^2} - \omega^2}\, \delta_{n,2} = 0$$

$$\Rightarrow \begin{cases} \alpha_n = 0 & \text{if } n \neq 2 \\ \alpha_2 = -\dfrac{1}{\frac{4c^2\pi^2}{L^2} - \omega^2} \end{cases}$$

$$u_t(x,0) = 0 \Rightarrow \sum_{n=1}^{\infty} A_n(x)\, \dot{B}_n(0) = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} A_n(x) \left[\beta_n \frac{n\pi c}{L} \right] = 0$$

$$\Rightarrow \beta_n = 0 \quad \forall n.$$