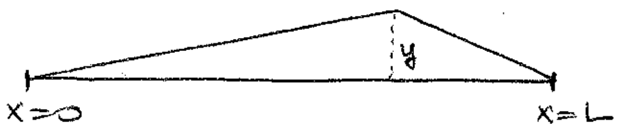
CHAPTER 4 Generic behaviour of Hyperbolic/ Parabolic/Elliptic equations through examples. Method of separation of variables

4.1 The wave equation: the vibrating string

Let a string be tightly stretched with ends attached at $x=0$ and $x=L$.



Displacements away from rest are measured by y.

The equation of motion of the string follows a wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

with boundary condition $y(0,t)=0$
$y(L,t)=0$.

Method of separation of variables

Let's assume that we can find a solution with the form

$$y(x,t) = X(x)T(t)$$

Then $$X(x)\frac{d^2T}{dt^2} = c^2 T(t)\frac{d^2X}{dx^2}$$

$$\Leftrightarrow \quad \underbrace{\frac{1}{T(t)}\frac{d^2T}{dt^2}}_{\text{a function of time only}} = \underbrace{\frac{c^2}{X(x)}\frac{d^2X}{dx^2}}_{\text{a function of x only}}.$$

This equality can only hold for all t, all x provided both sides are constant.

$$\frac{1}{T(t)}\frac{d^2T}{dt^2} = K \doteq \frac{c^2}{X(x)}\frac{d^2X}{dx^2}.$$

This constant is arbitrary for the moment, but it is an eigenvalue of the spatial operator $c^2\frac{d^2}{dx^2}$

$$c^2\frac{d^2X}{dx^2} = K\,X(x)$$

$\Rightarrow$ Will be determined by the boundary conditions.

We want to solve $\begin{cases} \dfrac{d^2X}{dx^2} = \dfrac{K}{c^2}X(x) \\ X(0) = X(L) = 0 \end{cases}$

if $K>0$ then the solutions are exponential
$K<0$ ——————— oscillatory.

if $K>0$ then it is not possible to fit both bcs unless $X(x)=0 \quad \forall x$

so let's choose $K<0$, and write $\frac{K}{c^2} = -k^2$

$$\Rightarrow \quad X(x) = A\cos kx + B\sin kx$$

$$X(0)=0 \Rightarrow A=0$$
$$X(L)=0 \Rightarrow \sin kL = 0 \Rightarrow k = \frac{n\pi}{L}$$

So there exist an ∞ of solutions for $X(x)$, namely

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right) \text{ with } n\in\mathbb{Z}$$

Now we must solve $\frac{d^2T}{dt^2} = KT(t) = -c^2k^2T(t)$

So $\quad T(t) = a\cos(kct) + b\sin(kct)$

since there are an ∞ of possible values of k then for each $X_n(x)$

corresponds a $T_n(t) = a_n \cos\left(\frac{n\pi c t}{L}\right) + b_n \sin\left(\frac{n\pi c t}{L}\right)$

$\Rightarrow$ The general solution to the wave equation with these boundary conditions is a linear combination of all the solutions:

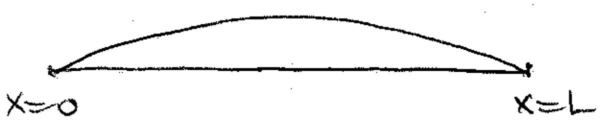
$$y(x,t) = \sum_n X_n(x) T_n(t) = \sum_n \sin\left(\frac{n\pi x}{L}\right)\left[\alpha_n \cos\left(\frac{n\pi c t}{L}\right) + \beta_n \sin\left(\frac{n\pi c t}{L}\right)\right]$$

(incorporate a_n with A_n and b_n with B_n + any linear combination coefficient).

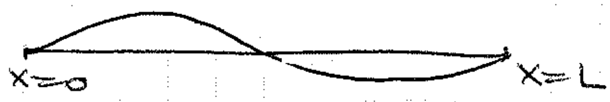
Note : • This expression shows that each spatial function $\sin\left(\frac{n\pi x}{L}\right)$ vibrates with its own frequency $\frac{nc}{2L}$ intrinsic to the system.

$\Rightarrow$ higher c, higher frenquency

$\Rightarrow$ longer L, lower frequency.



n=1
(0 nodes)



n=2
(1 node)

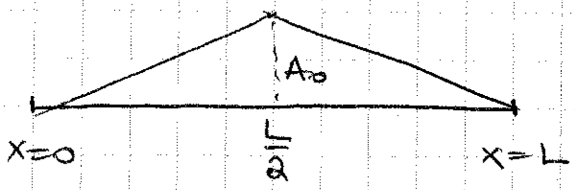
(4 nodes) n=5

$\Rightarrow$ Each mode has a different frequency. The higher the degree n of the mode, the higher the frequency (the pitch of the sound emitted for example).

• To determine which mode is excited and with which amplitude it is vibrating, we need to apply initial conditions to the system.

Example: Suppose we pluck the string in the middle, so that at $t=0$ we release it from rest with

$$y(x,0) = \begin{cases} \frac{2A_0}{L}x & \text{if } 0<x<\frac{L}{2} \\ 2A_0 - \frac{2x}{L}A_0 & \text{if } x\in\left[\frac{L}{2},L\right] \end{cases}$$



$$\frac{\partial y}{\partial t}(x,0)=0 \quad \text{since we release it from rest}$$

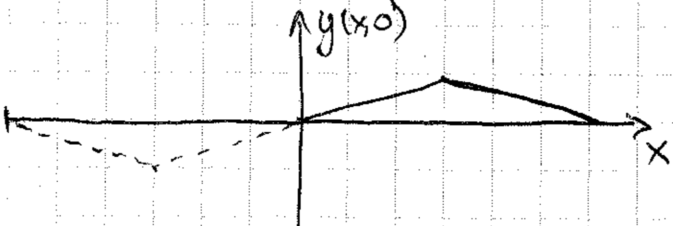
$\Rightarrow$ then we fit the general form $y(x,t)$ to these bcs by requiring that

$$y(x,0) = \sum_n{}' \alpha_n \sin\frac{n\pi x}{L} = \begin{cases} \frac{2A_0 x}{L} & \text{if } x\in\left[0,\frac{L}{2}\right] \\ 2A_0 - \frac{2A_0 x}{L} & \text{if } x\in\left[\frac{L}{2},L\right] \end{cases}$$

$$\frac{\partial y}{\partial t}(x,0) = \sum_n{}' \frac{n\pi c}{L}\beta_n \sin\left(\frac{n\pi x}{L}\right) = 0 \quad \Rightarrow \boxed{\beta_n=0}$$

The first of these two equations implies that we are seeking the coefficients α_n such that the series on the left is equal to the function on the RHS $\Rightarrow$ this looks like a Fourier series problem!

Problem: $y(x,0)$ is not periodic $\Rightarrow$ to remedy the problem, turn $y(x,0)$ into a periodic function by adding the interval $[-L,0]$



$\rightarrow$ construct an odd function since we are looking for a sin expansion

$\Rightarrow$ By definition $\alpha_n = \frac{1}{L}\int_{-L}^{L} y(x,0)\sin\left(\frac{n\pi x}{L}\right)dx$

by symmetry $= \frac{2}{L}\int_0^L y(x,0)\sin\left(\frac{n\pi x}{L}\right)dx$

$$\alpha_n = \frac{2}{L}\left\{ \int_0^{\frac{L}{2}} \left(\frac{2A_0 x}{L}\right) \sin\frac{n\pi x}{L}\, dx + \int_{\frac{L}{2}}^{L} \left(2A_0 - \frac{2A_0 x}{L}\right) \sin\frac{n\pi x}{L}\, dx \right\}$$

$$= \frac{2}{L}\left\{ \left[\frac{2A_0 x}{L}\left(-\frac{L}{n\pi}\right)\cos\frac{n\pi x}{L}\right]_0^{\frac{L}{2}} + \int_0^{\frac{L}{2}} \frac{2A_0}{L}\frac{L}{n\pi}\cos\frac{n\pi x}{L}\, dx \right.$$

$$+ 2A_0\left(-\frac{L}{n\pi}\right)\left[\cos\frac{n\pi x}{L}\right]_{\frac{L}{2}}^{L} - \left[\frac{2A_0 x}{L}\left(-\frac{L}{n\pi}\right)\cos\frac{n\pi x}{L}\right]_{\frac{L}{2}}^{L}$$

$$\left. - \int_{\frac{L}{2}}^{L} \frac{2A_0}{L}\frac{L}{n\pi}\cos\frac{n\pi x}{L}\, dx \right\}$$

$$= \frac{2}{L}\left\{ \left[-\frac{2A_0 x}{n\pi}\cos\frac{n\pi x}{L} + \frac{2A_0}{n\pi}\cdot\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right]_0^{\frac{L}{2}} \right.$$

$$\left. + \left[-\frac{2A_0 L}{n\pi}\cos\frac{n\pi x}{L} + \frac{2A_0 x}{n\pi}\cos\frac{n\pi x}{L} - \frac{2A_0}{n\pi}\cdot\frac{L}{n\pi}\sin\frac{n\pi x}{L}\right]_{\frac{L}{2}}^{L} \right\}$$

$$= \frac{2}{L}\left\{ \cancel{-\frac{2A_0}{n\pi}\frac{L}{2}\cos\left(\frac{n\pi}{2}\right)} + \frac{2A_0 L}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) \right.$$

$$\cancel{-\frac{2A_0 L}{n\pi}\cos(n\pi)} + \cancel{\frac{2A_0 L}{n\pi}\cos\left(\frac{n\pi}{2}\right)} + \cancel{\frac{2A_0 L}{n\pi}\cos(n\pi)}$$

$$\left. \cancel{-\frac{2A_0}{n\pi}\frac{L}{2}\cos\left(\frac{n\pi}{2}\right)} - \cancel{\frac{2A_0 L}{n^2\pi^2}\sin(n\pi)} + \frac{2A_0 L}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) \right]$$

$$= \frac{2}{L}\left\{ \frac{4A_0 L}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right\} = \frac{8A_0}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) \qquad |n| > 1$$

$\Rightarrow$ Finally,

$$y(x,t) = \sum_{n=1}^{\infty}{}' \underbrace{\frac{8A_0}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)}_{\text{amplitude of the mode}} \underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{\text{spatial mode}} \underbrace{\cos\left(\frac{n\pi c t}{L}\right)}_{\text{temporal variation of the mode.}}$$

(see movie)

The solution is a superposition of all the modes vibrating independently with constant amplitude determined by their initial conditions.

Note 1: The final solution is, in this case, a function that is also periodic in time, with the period $\frac{2L}{c}$. This is most easily seen by "re-thinking" the expression for $y(x,t)$ as a Fourier Series in time. This corresponds to the period of the fundamental mode ($n=1$).

→ Here we have a standing wave

Note 2 An interesting property of standing waves is that it can also be viewed as the superposition of two travelling waves with the same amplitude but opposite directions.

For example: For a single mode

$$\sin\left(\frac{n\pi x}{L}\right)\cdot\cos\left(\frac{n\pi ct}{L}\right) = \frac{1}{2}\Big(\underbrace{\sin\left(\frac{n\pi}{L}(x-ct)\right)}_{\text{a wave which travels to the right}} + \underbrace{\sin\left(\frac{n\pi}{L}(x+ct)\right)}_{\text{a wave which travels to the left.}}\Big)$$

(see movie).

For a sum of modes, we can rewrite

$$y(x,t) = \sum a_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)$$

$$= \sum \frac{a_n}{2}\sin\left(\frac{n\pi}{L}(x-ct)\right) + \frac{a_n}{2}\sin\left(\frac{n\pi}{L}(x+ct)\right)$$

(see movie) $= f(x-ct) + g(x+ct)$ (i.e, left & right "travelling" functions)

provided f and g are periodic functions of their respective variables $(x-ct)$ and $(x+ct)$. This is more subtle, (see later) but it works!

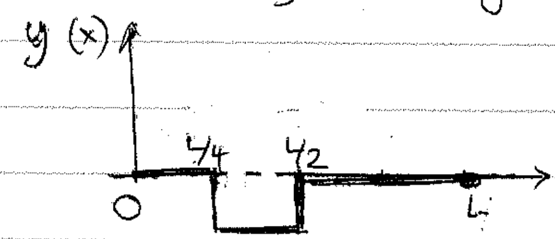
Example 2 : The "travelling" nature of the solutions is better illustrated in the following example. Let's consider the same overall problem but with different initial conditions.

Let

$$y(x,0) = \begin{cases} -1 & \text{if } x \in [\frac{L}{4}, L/2] \\ 0 & \text{otherwise} \end{cases}$$

$$y_t(x,0) = 0$$

(This could correspond to hitting it with a hammer and giving the string a velocity in that x-interval)



This time we set

$$\begin{cases} y_t(x,0) = 0 \\ y(x,0) = \sum \frac{n\pi c}{L} \beta_n \sin\left(\frac{n\pi x}{L}\right) = \begin{cases} -1 & \text{if } x \in [L/4, L/2] \\ 0 & \text{otherwise} \end{cases} \end{cases}$$

This implies $\beta_n = 0$. To find α_n, we construct (this time in our heads) the 2L periodic, odd extension of the function $y(x,0)$, let's call it $\tilde{y}(x,0)$. Then we know that

$$\alpha_n = \frac{1}{L} \int_{-L}^{L} \tilde{y}(x,0) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{L} \int_0^L y(x,0) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{L} \int_{L/4}^{L/2} - \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{L}\frac{L}{n\pi}\left[\cos\left(\frac{n\pi x}{L}\right)\right]_{L/4}^{L/2}$$

$$= \frac{2}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right) - \cos\left(\frac{n\pi}{4}\right)\right)$$

So $$y(x,t) = \sum_n \frac{2}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right) - \cos\left(\frac{n\pi}{4}\right)\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)$$

This corresponds to two propagating waves of opposite directions

$$y(x,t) = \sum_n \frac{1}{n\pi}\left(\cos\frac{n\pi}{2} - \cos\left(\frac{n\pi}{4}\right)\right)\sin\left(\frac{n\pi}{L}(x-ct)\right)$$
$$+ \sum_n \frac{1}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right) - \cos\left(\frac{n\pi}{4}\right)\right)\sin\left(\frac{n\pi}{L}(x+ct)\right)$$

<u>Note</u>. What would the solution have been if instead
$$\begin{cases} y(x,0) = 0 \\ y_t(x,0) = \begin{cases} -1 & \text{if } x \in [L/4, L/2] \\ 0 & \text{otherwise?} \end{cases} \end{cases}$$